Problem set 1
Math 207A, Fall 2011
Due: Wed., Oct. 5

1. Write the IVP for the forced, damped pendulum

\[ x_{tt} + \delta x_t + \omega_0^2 \sin x = \gamma \cos \omega t, \]
\[ x(0) = x_0, \quad x_t(0) = v_0 \]

as an IVP for an autonomous first-order system. What is the dimension of
the system?

2. Solve the scalar IVP

\[ x_t = x(\log x)^\alpha, \quad x(0) = x_0 \]

where \( \alpha > 0 \) and \( x_0 > 1 \). Find the maximal time-interval on which the
solution exists. For what values of \( \alpha \) does the solution exist for all times?

3. The position \( x(t) \in \mathbb{R} \) of a particle of mass \( m \) moving in one space
dimension in a potential \( V(x) \) satisfies

\[ mx_{tt} = -V'(x) \]

where the prime denotes a derivative with respect to \( x \). Show that the total
energy

\[ \frac{1}{2}mx_t^2 + V(x) = \text{constant} \]

is conserved. What can you say about the time-interval of existence of solutions for:
(a) the attractive potential \( V(x) = x^4 \); (b) the repulsive potential \( V(x) = -x^4 \)?

4. Linearize the Lorenz equations

\[ x_t = \sigma(y - x), \]
\[ y_t = rx - y - xz, \]
\[ z_t = xy - \beta z \]

about the equilibrium solution \((x, y, z) = (0, 0, 0)\). Show that this equilibrium
is linearly stable if \( r < 1 \) and linearly unstable if \( r > 1 \).