Problem set 1
Math 207A, Fall 2011
Solutions

1. Write the IVP for the forced, damped pendulum

\[ x_{tt} + \delta x_t + \omega_0^2 \sin x = \gamma \cos \omega t, \]
\[ x(0) = x_0, \quad x_t(0) = v_0 \]

as an IVP for an autonomous first-order system. What is the dimension of the system?

Solution

- Introduce variables \((x, v, s)\) where \(v = x_t, s = t\). Then

\[ x_t = v, \]
\[ v_t = -\delta v - \omega_0^2 \sin x + \gamma \cos \omega s, \]
\[ s_t = 1, \]

with initial conditions

\[ x(0) = x_0, \quad v(0) = v_0, \quad s(0) = 0. \]

Remark. Since the phase space is three-dimensional, chaotic behavior is possible, and indeed it occurs in suitable parameter regimes.
2. Solve the scalar IVP

\[ x_t = x(t)(\log x)^\alpha, \quad x(0) = x_0 \]

where \( \alpha > 0 \) and \( x_0 > 1 \). Find the maximal time-interval on which the solution exists. For what values of \( \alpha \) does the solution exist for all times?

**Solution**

- Separating variables, we get

\[ \int \frac{1}{x(t)(\log x)^\alpha} \, dx = \int \, dt \]

To compute the \( x \)-integral, use the substitution \( u = \log x \), which gives

\[ \int \frac{1}{u^\alpha} \, du = t + C \]

where \( C \) is a constant of integration.

- If \( \alpha \neq 1 \), the solution is

\[ \frac{1}{1 - \alpha} u^{1-\alpha} = t + C. \]

The initial condition implies that

\[ C = \frac{1}{1 - \alpha} (\log x_0)^{1-\alpha}, \]

so

\[ x(t) = \exp\left\{\left[ (1 - \alpha)t + (\log x_0)^{1-\alpha}\right]^{1/(1-\alpha)}\right\}. \]

Note that \( \log x_0 > 0 \) since \( x_0 > 1 \).

- If \( 0 < \alpha < 1 \), the solution exists for all \( 0 \leq t < \infty \).

- If \( \alpha > 1 \) the solution exists only for a finite time interval \( 0 \leq t < T \) where \( T > 0 \) is given by

\[ T = \frac{1}{(\alpha - 1)(\log x_0)^{\alpha-1}}. \]
If \( \alpha = 1 \), the solution is

\[
x(t) = \exp \left[ (\log x_0) \exp t \right].
\]

The solution exist for all \( t \), although it grows very rapidly (doubly exponentially) as \( t \to \infty \).

**Remark.** Note that the solution of \( x_t = x \log x \) exists globally in time even though the right-hand side grows faster than a linear function of \( x \), albeit by a slowly growing logarithmic factor. Any higher power of \( \log x \), however, leads to solutions that blow up in finite time.
3. The position \( x(t) \in \mathbb{R} \) of a particle of mass \( m \) moving in one space dimension in a potential \( V(x) \) satisfies

\[
mx_{tt} = -V'(x)
\]

where the prime denotes a derivative with respect to \( x \). Show that the total energy

\[
\frac{1}{2}mx^2_t + V(x) = \text{constant}
\]

is conserved. What can you say about the time-interval of existence of solutions for: (a) the attractive potential \( V(x) = x^4 \); (b) the repulsive potential \( V(x) = -x^4 \)?

Solution

- Using the chain rule and the ODE, we get

\[
\frac{d}{dt} \left[ \frac{1}{2}mx^2_t + V(x) \right] = mx_tx_{tt} + V'(x)x_t \\
= -x_tV'(x) + V'(x)x_t \\
= 0.
\]

Hence the total energy is constant.

- If \( V(x) = x^4 \) then

\[
\frac{1}{2}mx^2_t + x^4 = \text{constant},
\]

which implies that both \( x, x_t \) are bounded functions of time. The extension theorem, applied to the corresponding first-order systems for \((x, x_t)\), then implies that the solutions exist globally for all \( t \in \mathbb{R} \).

- In fact, if \( V(x) = x^4 \), all non-zero solutions are periodic functions of \( t \) (as is, strictly speaking, the zero solution).

- If \( V(x) = -x^4 \) then

\[
\frac{1}{2}mx^2_t - x^4 = \text{constant},
\]

but this does not imply that \( x, x_t \) remain bounded, so there is no conclusion from the extension theorem.
• In fact, if $V(x) = -x^4$ and $\frac{1}{2}mx^2_t - x^4 = E_0$, where

$$E_0 = \frac{1}{2}mv_0^2 - x_0^4, \quad x(0) = x_0, \quad x_t(0) = v_0,$$

then $x(t)$ satisfies the first order ODE

$$x_t = \pm \sqrt{\frac{2}{m}(E_0 + x^4)}, \quad x(0) = x_0$$

with an appropriate choice of the sign.

• If $E_0 \neq 0$, then solutions of this ODE (whose right hand side grows like $x^2$) go off to infinity in finite time both as $t \to -\infty$ and $t \to \infty$. The (unstable) equilibrium solution $x(t) = 0$ exists for all time. Finally, if $E_0 = 0$ and $x(t) \neq 0$, then: when $x_0, v_0$ have the same sign, solutions go off to infinity in finite time as $t \to \infty$ and approach 0 as $t \to -\infty$; when $x_0, v_0$ have the opposite sign, solutions approach 0 as $t \to \infty$ and blow up at finite negative time.

**Remark.** The previous statements may be easier to follow if you sketch the $(x, x_t)$-phase plane of the system (as we’ll do in class later on).
4. Linearize the Lorenz equations

\[ x_t = \sigma(y - x), \]
\[ y_t = rx - y - xz, \]
\[ z_t = xy - \beta z \]

about the equilibrium solution \((x, y, z) = (0, 0, 0)\). Show that this equilibrium is linearly stable if \(r < 1\) and linearly unstable if \(r > 1\).

Solution

- We obtain the linearized system is by neglecting the quadratically non-linear terms, which gives

\[ x_t = \sigma(y - x), \]
\[ y_t = rx - y, \]
\[ z_t = -\beta z. \]

In matrix form, this system is

\[ \vec{x}_t = A\vec{x} \]

where \(\vec{x} = (x, y, z)^T\) and

\[
A = \begin{pmatrix}
-\sigma & \sigma & 0 \\
r & -1 & 0 \\
0 & 0 & -\beta \\
\end{pmatrix}.
\]

- The equilibrium \((0, 0, 0)^T\) is stable if all eigenvalues of \(A\) have negative real part and unstable if some eigenvalue of \(A\) has positive real part.

- We have

\[
\det(A - \lambda I) = \begin{vmatrix}
-\lambda - \sigma & \sigma & 0 \\
r & -\lambda - 1 & 0 \\
0 & 0 & -\lambda - \beta \\
\end{vmatrix} = -(\lambda + \beta) \left[ \lambda^2 + (\sigma + 1)\lambda + (1 - r)\sigma \right].
\]

Hence, the eigenvalues of \(A\) are

\[
\lambda = -\beta, \quad \lambda = \frac{1}{2} \left[ -(\sigma + 1) \pm \sqrt{(\sigma + 1)^2 - 4(1 - r)\sigma} \right].
\]
• We assume that the parameters $\sigma$, $r$, $\beta$ are positive. Then $\lambda = -\beta < 0$ is a stable eigenvalue.

• Suppose that $r < 1$.
  
  - If $4(1 - r)\sigma > (\sigma + 1)^2$, then the remaining eigenvalues
    \[ \lambda = -\frac{1}{2} [(\sigma + 1) \pm i\alpha], \quad \alpha = \sqrt{4\sigma(1 - r) - (\sigma + 1)^2} \]
    are complex with negative real part.
  
  - If $0 \leq 4(1 - r)\sigma < (\sigma + 1)^2$, then
    \[ 0 \leq \sqrt{(\sigma + 1)^2 - 4(1 - r)\sigma} < \sigma + 1, \]
    and the remaining eigenvalues are real and negative.

In either case, the equilibrium $(0, 0, 0)^T$ is stable.

• Suppose that $r > 1$. Then
  \[ \sqrt{(\sigma + 1)^2 - 4(1 - r)\sigma} > \sigma + 1 \]
  and therefore
  \[ \lambda = \frac{1}{2} \left[ -(\sigma + 1) + \sqrt{(\sigma + 1)^2 - 4(1 - r)\sigma} \right] > 0 \]
  so the equilibrium $(0, 0, 0)^T$ is unstable.

Remark. In the context of the Lorenz equation as a model of a fluid layer heated from below, this result has the interpretation that when the temperature difference (proportional to the Rayleigh number $r$) is sufficiently small, then a stationary equilibrium in which the fluid is a rest and transfers heat from bottom to top by conduction is stable. But when the temperature difference is too large, this equilibrium becomes unstable, leading to a convective motion.