Problem set 4
Math 207A, Fall 2011
Solutions

1. Newton’s method for the iterative solution of the scalar equation \( f(x) = 0 \) is

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.
\]

If \( f(x) = x^2 - 2 \), show that this equation becomes

\[
x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}.
\]

What are the fixed points of this system? Determine their stability. Compute \( x_4 \) numerically if \( x_0 = 3 \).

Solution

- For \( f(x) = x^2 - 2 \), we have

\[
x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{x_n}{2} + \frac{1}{x_n}.
\]

- Fixed points \( x \) satisfy

\[
x = F(x), \quad F(x) = \frac{x}{2} + \frac{1}{x},
\]

which implies that \( x^2 = 2 \) or \( x = \pm \sqrt{2} \).

- We have

\[
F'(x) = \frac{1}{2} - \frac{1}{x^2},
\]

so \( F'(\pm 1/\sqrt{2}) = 0 \). Since this has absolute value less than one, both fixed points are asymptotically stable. In fact, since \( F' = 0 \) at the fixed points, we get very rapid, quadratically nonlinear convergence of nearby iterates to the fixed point.
• We have

\[ F(3) = 1.833333\ldots, \]
\[ F(1.833333\ldots) = 1.462121\ldots, \]
\[ F(1.462121\ldots) = 1.414998\ldots, \]
\[ F(1.414998\ldots) = 1.41421378\ldots. \]

The forth iterate already agrees to six decimal places with \( \sqrt{2} = 1.41421356\ldots. \)
2. Find the fixed points of the system

\[ x_{n+1} = -\frac{\mu}{2} \tan^{-1} x_n \]

and determine their stability. Show that a period-doubling bifurcation occurs at \( \mu = 2 \). Is the resulting period-two orbit stable or unstable?

**Solution**

- The fixed points satisfy

\[ x = -\frac{\mu}{2} \tan^{-1} x \]

One solution branch is \( x = 0 \), defined for all values of \( \mu \). We can’t solve for the other fixed points explicitly, but by looking at the intersection of the graphs

\[ y = -\frac{2}{\mu} x, \quad y = \tan^{-1} x \]

we see that there are two other fixed points \( x = \pm \bar{x}(\mu) \) for \( \mu < -2 \), where \( \bar{x}(\mu) \to \pi/2 \) as \( \mu \to -\infty \). For \( \mu \geq -2 \), \( x = 0 \) is the only fixed point.

- Writing the equation as \( x_{n+1} = f(x_n; \mu) \) where

\[ f(x; \mu) = -\frac{\mu}{2} \tan^{-1} x, \]

we have

\[ f_x(x; \mu) = -\frac{\mu}{2} \frac{1}{1 + x^2}. \]

- We have

\[ f_x(0; \mu) = -\frac{\mu}{2} \]

so \( x = 0 \) is asymptotically stable for \( |\mu| < 2 \) and unstable if \( |\mu| > 2 \).

- To determine the stability of the other fixed points (which have the same stability since \( f \) is an odd function of \( x \)) note that

\[ f_x(\bar{x}(\mu); \mu) \to \infty \quad \text{as} \quad \mu \to -\infty. \]

One can check that \( f(\bar{x}, \mu) \neq 1 \) for \( \mu < -2 \), so \( f_x(\bar{x}, \mu) > 1 \) by continuity. The fixed points are therefore unstable.
• The multiplier $f_x(0;\mu)$ passes through 1 at $\mu = -2$, and there is a subcritical pitchfork bifurcation of fixed points at $(x,\mu) = (0,-2)$, as shown by the previous solution for the fixed points of $f$.

• To determine the local behavior of the bifurcation at $(x,\mu) = (0,2)$, we Taylor expand about this point. Writing $\mu = 2 + \mu_1$, we get

$$f(x,\mu) = -\left(1 + \frac{1}{2}\mu_1\right) \left(x - \frac{1}{3}x^3 + \ldots\right)$$

$$= -\left(1 + \frac{1}{2}\mu_1\right)x + \frac{1}{3}x^3 + \ldots$$

and

$$f^2(x,\mu) = -\left(1 + \frac{1}{2}\mu_1\right) \left[-\left(1 + \frac{1}{2}\mu_1\right)x + \frac{1}{3}x^3 + \ldots\right]$$

$$+ \frac{1}{3} \left[-\left(1 + \frac{1}{2}\mu_1\right)x + \frac{1}{3}x^3 + \ldots\right]^3 + \ldots$$

$$= (1 + \mu_1)x - \frac{2}{3}x^3 + \ldots$$

• Thus $f^2$ has a supercritical pitchfork bifurcation at $(0,2)$, and $f$ has a supercritical period-doubling bifurcation in which a stable periodic orbit appears above $\mu = 2$. The points on the orbit are fixed points of $f^2$; they are given approximately by

$$x = \pm \sqrt[3]{\frac{3(\mu - 2)}{2}} + \ldots$$
3. Consider the discrete dynamical system on the circle for \( x_n \in \mathbb{T} \)

\[
x_{n+1} = x_n + \mu \pmod{2\pi}
\]

corresponding to rotation by an angle \( \mu \in \mathbb{T} \). Describe the structure of the orbits and how they depend on \( \mu \).

**Solution**

- The orbit structure depends on whether \( \mu \) is a rational or irrational multiple of \( 2\pi \).

- If

\[
\frac{\mu}{2\pi} = \frac{p}{q} \in \mathbb{Q}
\]

where \( p, q \) are relatively prime, then every orbit is periodic with minimal period \( q \).

- If

\[
\frac{\mu}{2\pi} \notin \mathbb{Q}
\]

then every orbit consists of a countably infinite sequence of distinct points. One can show that every orbit is dense in \( \mathbb{T} \) and equidistributed (Weyl's theorem).
4. Carry out numerical experiments for iterations of the logistic map

\[ x_{n+1} = \mu x_n (1 - x_n) \]

where \( 1 \leq \mu \leq 4 \) and \( 0 \leq x_0 \leq 1 \). (You can write your own program or use the MATLAB script provided on the course website.)

**Solution**

- You should see a sequence of period doubling bifurcations, followed by chaotic behavior. Inside the chaotic region, there are ‘windows’ with stable period 3 orbits. See the text for further details.