1. Mathieu’s equation for $x(t) \in \mathbb{R}$ is
   \[ x_{tt} + (a - 2q \cos 2t) x = 0, \]
   where $a, q$ are constant parameters. This ODE describes a simple-harmonic oscillator whose frequency varies periodically in time. Write Mathieu’s equation as a first-order autonomous system. Is Mathieu’s equation linear? Is the corresponding autonomous first-order system linear?

2. Solve the IVP with $x(0) = x_0$ for the following scalar ODEs:
   (a) $x_t = x^{1/3}$; (b) $x_t = x^3$; (c) $x_t = \frac{x^3}{1 + x^2}$.
   Discuss the existence (local/global) and uniqueness of solutions in each case.

3. A gradient system for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is a system of the form
   \[ \dot{x} = -\nabla V(x), \quad \text{or} \quad \dot{x}_i = -\frac{\partial V}{\partial x_i} \quad (1 \leq i \leq n), \]
   where $V : \mathbb{R}^n \to \mathbb{R}$ is a smooth function and $\nabla$ is the gradient with respect to $x$.
   (a) If $x(t)$ is a solution of this gradient system with $x(0) = x_0$, show that $V(x(t)) \leq V(x_0)$ for all $t \geq 0$.
   (b) Show that the following system for $(x, y) \in \mathbb{R}^2$ is a gradient system
   \[ \dot{x} = -x + 2y - x^3, \quad \dot{y} = 2x - y - y^3, \]
   and deduce that solutions of the initial value problem exist for all $t \geq 0$. Do solutions necessarily exist for all $t < 0$?

4. Write a MATLAB script to solve the Lorentz equations
   \[ \dot{x} = s(-x + y), \quad \dot{y} = rx - y - xz, \quad \dot{z} = xy - bz, \]
   with initial conditions $x(0) = x_0, y(0) = y_0, z(0) = z_0$. Use Lorenz’s parameter values $s = 10, r = 28, b = 8/3$ to compute the following solutions. Submit a copy of your script and the two plots.
   (a) Plot the trajectory for initial data $(x_0, y_0, z_0) = (0, 1, 0)$ as a parametric curve in $(x, y, z)$-phase space for $0 \leq t \leq 30$.
   (b) Plot, on the same graph, the solutions for $x(t)$ with $0 \leq t \leq 30$ and the two sets of initial data: (i) $(x_0, y_0, z_0) = (0, 1, 0)$; (ii) $(x_0, y_0, z_0) = (0, 1.01, 0)$.