Problem set 1
Math 207A, Fall 2018
Due: Fri., Oct. 5

1. Consider the ODE for \(x(t) \in \mathbb{R}\) given by

\[
\dot{x} = x \log |x|.
\]

(a) Compute the flow map \(\varphi_t : \mathbb{R} \to \mathbb{R}\). (We define \(x \log |x| = 0\) for \(x = 0\).)
(b) Use your solution in (a) to verify explicitly that \(\varphi_t\) satisfies the group property \(\varphi_s \circ \varphi_t = \varphi_{s+t}\), and find the fixed points of \(\varphi_t\).

2. Define \(E : \mathbb{R}^2 \to \mathbb{R}\) by

\[
E(x, y) = \frac{1}{6} x^3 - \frac{1}{2} x^2 + \frac{1}{2} y^2.
\]

(a) Sketch the phase plane of the Hamiltonian system

\[
\dot{x} = \frac{\partial E}{\partial y}, \quad \dot{y} = -\frac{\partial E}{\partial x},
\]

and discuss the stability of the equilibria.

(a) Sketch the phase plane of the gradient system

\[
\dot{x} = -\frac{\partial E}{\partial x}, \quad \dot{y} = -\frac{\partial E}{\partial y},
\]

and discuss the stability of the equilibria.

3. The following Hamiltonian, depending on \((q_1, q_2, p_1, p_2) \in \mathbb{R}^4\), describes two decoupled simple harmonic oscillators, one with positive energy, the other with negative energy:

\[
H(q_1, q_2, p_1, p_2) = \frac{1}{2} \left( q_1^2 + p_1^2 \right) - \frac{1}{2} \left( q_2^2 + p_2^2 \right).
\]

(a) Write down Hamilton’s equations and solve them. Deduce that the equilibrium \((q_1, q_2, p_1, p_2) = (0, 0, 0, 0)\) is stable. What kind of critical point does \(H\) have at this equilibrium?
(b) Suppose we include an interaction term in the Hamiltonian
\[
H(q_1, q_2, p_1, p_2) = \frac{1}{2} (q_1^2 + p_1^2) - \frac{1}{2} (q_2^2 + p_2^2) + kq_1q_2,
\]
where \( k \in \mathbb{R} \) is a constant. What happens to the stability of the equilibrium?

4. Consider the Lorenz equations
\[
\begin{align*}
    x_t &= \sigma(y - x), \\
    y_t &= rx - y - xz, \\
    z_t &= xy - \beta z,
\end{align*}
\]
with parameter values \( \sigma = 10, \beta = 8/3, r = 28 \).

(a) Solve the Lorenz equations numerically with initial conditions
\[
    x(0) = -2, \quad y(0) = -4, \quad z(0) = 12
\]
for \( 0 \leq t \leq 30 \). Plot the trajectory of this solution in \((x, y, z)\)-phase space, and plot the graph of \( x(t) \) versus \( t \).

(b) Solve the Lorenz equations numerically with initial conditions
\[
    x(0) = -2.0001, \quad y(0) = -4, \quad z(0) = 12
\]
for \( 0 \leq t \leq 30 \), and plot the graph of \( x(t) \) versus \( t \) on the same plot as the one from (a).

HINT. In MATLAB, use \texttt{ode45} to solve the ODE, \texttt{plot3} to plot the trajectory in phase space, and \texttt{plot} to plot the graphs of \( x(t) \).