1. A model for a population $x(t) \geq 0$ with logistic growth and a constant rate of harvesting is

$$x_t = \mu x \left(1 - \frac{x}{K}\right) - H$$

where the parameters $\mu, K, H$ are positive constants.

(a) Show that a nondimensionalized form of the equation is

$$x_t = x(1 - x) - h,$$

and express the dimensionless parameter $h$ as a ratio of two times.

(b) Sketch a graph of the equilibria as functions of $h$, and sketch the phase line of (1) for various values of $h > 0$. Determine the stability of the equilibria, both from the phase line and from their linearized stability. For what values of the initial (nondimensionalized) population $x_0 > 0$ and harvesting rate $h > 0$ does the population become extinct?

2. The graph $y = f(x)$ of a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ is shown below. The function $f$ has zeros only at certain integer values of $x$ and is never zero outside the $x$-interval shown.

(a) Sketch the phase line for the ODE $x_t = f(x)$ and state the stability of the equilibria. Which of the equilibria are hyperbolic?

(b) Sketch the graph of the solution of the initial value problem with $x(0) = 0$.

(c) Sketch the graph of a potential $E(x)$ such that $f(x) = -E'(x)$. 

![Graph of f(x)]
3. A spherical raindrop with volume $V(t)$ and surface area $A(t)$ evaporates at a rate proportional to its surface area, meaning that $V_t = -kA$ for some constant $k > 0$. Write down an ODE for $V$ and show that the raindrop evaporates completely in finite time. Find an expression for the evaporation time $T$ in terms of $k$ and the initial volume $V_0$ of the drop, and verify that your result is dimensionally consistent. Why doesn’t this result violate the uniqueness part of the Picard theorem?

4. (a) Consider a scalar ODE $x_t = f(x)$ where $f : \mathbb{R} \to \mathbb{R}$ is continuous. Prove that the ODE cannot have a non-constant periodic solution with minimal period $T > 0$ such that $x(t + T) = x(t)$ for all $t \in \mathbb{R}$. HINT. Consider the integral

$$\int_0^T f(x) x_t dt,$$

(b) Why doesn’t your argument in (a) apply to an ODE $\theta_t = f(\theta)$ on the circle $\mathbb{T}$?

5. (a) A Bernoulli equation is an ODE of the form

$$x_t = a(t)x + b(t)x^n$$

where $a$, $b$ are continuous functions and $n \neq 1$. Show that the transformation

$$u = \frac{1}{x^{n-1}}$$

reduces a Bernoulli equation to a linear equation for $u$. Use this transformation to solve the logistic equation $x_t = x(1 - x)$.

(b) A Riccati equation is an ODE of the form

$$x_t = a(t) + b(t)x + c(t)x^2.$$

where $a$, $b$, $c$ are continuous functions, with $c \neq 0$. Show that the transformation

$$x = -\frac{u_t}{cu}$$

reduces the Riccati equation to a second order, linear equation for $u$. Use this transformation to solve the logistic equation $x_t = x(1 - x)$. 