1. Find an explicit expression for the Green’s function for the problem
\[-u'' + u = f(x), \quad 0 < x < 1\]
\[u'(0) = 0, \quad u'(1) = 0.\]

Write down the Green’s function representation of the solution for \(u(x)\).

Solution

- Homogeneous solutions that satisfy the BCs at the left and right endpoints are
  \[u_1(x) = \cosh x, \quad u_2(x) = \cosh(1 - x)\]
  with Wronskian \(-\sinh 1\), so the Green’s function is
  \[G(x, \xi) = \begin{cases} 
  \cosh x \cosh(1 - \xi)/\sinh 1 & 0 \leq x \leq \xi, \\
  \cosh \xi \cosh(1 - x)/\sinh 1 & \xi \leq x \leq 1.
  \end{cases}\]

- The Green’s function representation of the solution is
  \[u(x) = \int_0^1 G(x, \xi) f(\xi) \, d\xi.\]
2. Use separation of variables and Fourier series to solve the following IBVP for the Schrödinger equation for the complex-valued function \( \psi(x, t) \)

\[
i \psi_t = -\psi_{xx}, \quad 0 < x < 1
\]

\[
\psi(0, t) = 0, \quad \psi(1, t) = 0,
\]

\[
\psi(x, 0) = f(x)
\]

where \( f \in L^2(0, 1) \) is given initial data. Show from your solution that

\[
\int_0^1 |\psi(x, t)|^2 \, dx = \int_0^1 |f(x)|^2 \, dx \quad \text{for all } t \in \mathbb{R}.
\]

Solution

- The solution is

\[
\psi(x, t) = \sum_{n=1}^{\infty} c_n e^{-in^2\pi^2 t} \sin(n\pi x)
\]

where

\[
c_n = 2 \int_0^1 f(x) \sin(n\pi x) \, dx.
\]

- By Parseval’s theorem, and the fact that \(|e^{-i\theta}| = 1\), we have for any \( t \in \mathbb{R} \) that

\[
\int_0^1 |\psi(x, t)|^2 \, dx = \frac{1}{2} \sum_{n=1}^{\infty} |c_n e^{-in^2\pi^2 t}|^2
\]

\[
= \frac{1}{2} \sum_{n=1}^{\infty} |c_n|^2
\]

\[
= \int_0^1 |f(x)|^2 \, dx.
\]
3. After non-dimensionalization, the displacement $u(x)$ of a non-uniform string, with density $\rho(x)$, fixed at each end and vibrating with frequency $\omega$ satisfies the EVP

$$- u'' = \lambda \rho(x) u, \quad 0 < x < 1,$$

$$u(0) = 0, \quad u(1) = 0$$

where $\lambda = \omega^2$. The fundamental frequency of the string is $\omega_1 = \sqrt{\lambda_1}$, where $\lambda = \lambda_1$ is the smallest eigenvalue. If $m \leq \rho(x) \leq M$ where $m, M$ are positive constants, show that

$$\frac{\pi}{\sqrt{M}} \leq \omega_1 \leq \frac{\pi}{\sqrt{m}}.$$

Does this result make sense physically?

**Solution**

- The Rayleigh quotient for the minimum eigenvalue is

$$\lambda_1 = \min_{u \neq 0} \frac{\int_0^1 u'(x)^2 \, dx}{\int_0^1 \rho(x)u(x)^2 \, dx}.$$

- The Rayleigh quotient for the minimum eigenvalue $\mu_1$ of the problem with constant density $\rho_0$

$$- u'' = \mu \rho_0 u, \quad 0 < x < 1,$$

$$u(0) = 0, \quad u(1) = 0$$

is

$$\mu_1 = \min_{u \neq 0} \frac{\int_0^1 u'(x)^2 \, dx}{\int_0^1 \rho_0 u(x)^2 \, dx}.$$

In this case, we have an explicit solution for the minimum eigenvalue

$$\mu_1 = \frac{\pi^2}{\rho_0}$$

with eigenfunction $\sin(\pi x)$. 
• If \( \rho(x) \geq m \) for all \( x \in [0, 1] \) then (taking \( \rho_0 = m \))
\[
\int_0^1 \rho(x)u(x)^2 \, dx \geq \int_0^1 mu(x)^2 \, dx
\]
for every function \( u(x) \), so
\[
\frac{\int_0^1 u'(x)^2 \, dx}{\int_0^1 \rho(x)u(x)^2 \, dx} \leq \frac{\int_0^1 u'(x)^2 \, dx}{\int_0^1 mu(x)^2 \, dx}.
\]
It follows that \( \lambda_1 \leq \mu_1 \), or
\[
\omega_1 \leq \frac{\pi}{\sqrt{m}}
\]

• If \( \rho(x) \leq M \) for all \( x \in [0, 1] \) then (taking \( \rho_0 = M \))
\[
\int_0^1 \rho(x)u(x)^2 \, dx \leq \int_0^1 Mu(x)^2 \, dx
\]
for every function \( u(x) \), so
\[
\frac{\int_0^1 u'(x)^2 \, dx}{\int_0^1 \rho(x)u(x)^2 \, dx} \geq \frac{\int_0^1 u'(x)^2 \, dx}{\int_0^1 Mu(x)^2 \, dx}.
\]
It follows that \( \lambda_1 \geq \mu_1 \), or
\[
\omega_1 \geq \frac{\pi}{\sqrt{M}}
\]

• The result states that the fundamental frequency of a nonuniform string
is greater than that of a heavier uniform string and less than that of a
lighter uniform string, which is what one would expect physically.
4. Consider the Volterra integral operator $K : L^2(0, 1) \to L^2(0, 1)$ defined by

$$Ku(x) = \int_0^x u(y) \, dy, \quad 0 < x < 1$$

Show that the integral equation $Ku = \lambda u$ has no nonzero solutions for any $\lambda \in \mathbb{C}$, meaning that $K$ has no eigenvalues. Why doesn’t this contradict the spectral theorem for compact (or Hilbert-Schmidt) self-adjoint operators?

**Solution**

- If $\lambda = 0$, then $Ku = 0$ and
  
  $$u = (Ku)' = 0,$$

  so $0$ is not an eigenvalue of $K$.

- If $\lambda \neq 0$, then differentiating the equation $Ku = \lambda u$, and also setting $x = 0$ in the integral equation, we get
  
  $$\lambda u' = u, \quad u(0) = 0.$$

  The general solution of the ODE is
  
  $$u(x) = ce^{x/\lambda}.$$

  The IC implies that $c = 0$, so $u = 0$ and $\lambda$ is not an eigenvalue of $K$.

- The operator $K$ is Hilbert-Schmidt, but it is not self-adjoint on $L^2(0, 1)$. In fact, its adjoint is
  
  $$(K^* u)(x) = \int_x^1 u(y) \, dy$$

  (Moral: The spectral theory of non-self-adjoint operators is not nearly as nice as the theory for self-adjoint operators.)
5. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded region, and define an operator $L$ by

$$Lu = -\nabla \cdot (p \nabla u) + qu$$

where $p, q$ are smooth functions on $\bar{\Omega}$. Show that

$$\int_{\Omega} uLv \, dx = \int_{\Omega} vLu \, dx$$

for all functions $u, v : \Omega \to \mathbb{R}$ that vanish on the boundary $\partial \Omega$, meaning that $L$ with Dirichlet BCs is formally self-adjoint.

**Solution**

- We have the identity

$$u \nabla \cdot (p \nabla v) - v \nabla \cdot (p \nabla u) = \nabla \cdot (puv - pvu).$$

To show this, we compute in Cartesian components (using the summation convention) that

$$\nabla \cdot (puv - pvu) = \frac{\partial}{\partial x_i} \left( pu \frac{\partial v}{\partial x_i} - pv \frac{\partial u}{\partial x_i} \right)$$

$$= u \frac{\partial}{\partial x_i} \left( p \frac{\partial v}{\partial x_i} \right) - v \frac{\partial}{\partial x_i} \left( p \frac{\partial u}{\partial x_i} \right)$$

$$= u \nabla \cdot (p \nabla v) - v \nabla \cdot (p \nabla u).$$

- Using this identity and the divergence theorem, we get

$$\int_{\Omega} (uLv - vLu) \, dx = \int_{\Omega} \{-u \nabla \cdot (p \nabla v) + qw + v \nabla \cdot (p \nabla u) - qvu \} \, dx$$

$$= -\int_{\Omega} \{u \nabla \cdot (p \nabla v) - v \nabla \cdot (p \nabla u)\} \, dx$$

$$= -\int_{\Omega} \nabla \cdot (puv - pvu) \, dx$$

$$= -\int_{\partial \Omega} \left( pu \frac{\partial v}{\partial n} - pv \frac{\partial u}{\partial n} \right) \, dS.$$
The integral over the boundary vanishes since $u, v = 0$ on $\partial \Omega$, so

$$\int_{\Omega} uLv \, dx = \int_{\Omega} vLu \, dx$$

- This is a multi-dimensional analog of the corresponding self-adjointness identity for the one-dimensional Sturm-Liouville operator

$$Lu = -(pu')' + qu$$

with Dirichlet BCs.
6. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded region, Consider the Neumann BVP

$$\begin{aligned}
-\Delta u &= f(x) \quad x \in \Omega, \\
\frac{\partial u}{\partial n} &= g(x) \quad x \in \partial \Omega.
\end{aligned}$$

(a) Show that a solution can only exist if

$$\int_{\Omega} f \, dx + \int_{\partial \Omega} g \, dS = 0$$

Give a physical interpretation of this result in terms of heat flow.

(b) If a solution exists, show that it is unique up to an arbitrary additive constant.

Solution

- (a) Assume there is a solution $u$. Then, using the divergence theorem, we get

$$\int_{\Omega} f \, dx = -\int_{\Omega} \Delta u \, dx = -\int_{\partial \Omega} \frac{\partial u}{\partial n} \, dS = -\int_{\partial \Omega} g \, dS$$

which gives the result.

- The problem describes the equilibrium temperature of a body with heat source density $f$ and prescribed heat flux $g$ into the body through its boundary. An equilibrium solution is only possible if the net rate at which internal sources generate heat ($\int_{\Omega} f \, dx$) is equal to the heat flux out of the body ($-\int_{\partial \Omega} g \, dS$).

- (b) Suppose $u_1, u_2$ are two solutions, and let $v = u_1 - u_2$. Then, by linearity,

$$\begin{aligned}
\Delta v &= 0 \quad x \in \Omega, \\
\frac{\partial v}{\partial n} &= 0 \quad x \in \partial \Omega.
\end{aligned}$$

- According to Green’s first identity

$$\int_{\Omega} \left( v \Delta v + |\nabla v|^2 \right) \, dx = \int_{\Omega} \nabla \cdot (v \nabla v) \, dx = \int_{\partial \Omega} v \frac{\partial v}{\partial n} \, dS.$$
Using the equations for $v$ in this identity, we get that

$$
\int_{\Omega} |\nabla v|^2 \, dx = 0.
$$

It follows that $\nabla v = 0$ in $\Omega$, meaning that $v$ is constant, so any two solutions are equal up to a constant.