1. If $\lambda \leq 0$, show that there are no non-zero solutions for $u(x)$ of the Sturm-Liouville problem

$$-u'' = \lambda u \quad 0 < x < 1,$$

$$u(0) = 0, \quad u(1) = 0.$$

**Solution**

- If $\lambda < 0$, with $\lambda = -k^2$ for $k > 0$ say, then the general solution of the ODE is

$$u(x) = c_1 \cosh kx + c_2 \sinh kx.$$ 

The BC $u(0) = 0$ implies that $c_1 = 0$, and then the BC $u(1) = 0$ implies that $c_2 = 0$, since $\sinh k \neq 0$. Therefore $u = 0$ and $\lambda$ is not an eigenvalue.

- If $\lambda = 0$, then the general solution of the ODE is

$$u(x) = c_1 + c_2 x$$

and the BCs imply that $c_1 = c_2 = 0$, so $\lambda = 0$ is not an eigenvalue.
2. Find all eigenvalues $\lambda_n$ and eigenfunctions $u_n(x)$ of the Sturm-Liouville problem

$$-u'' = \lambda u, \quad 0 < x < 1,$$

$$u'(0) = 0, \quad u'(1) = 0$$

with Neumann boundary conditions. Show that

$$\int_0^1 u_n(x)u_m(x) \, dx = 0$$

if $n \neq m$.

Solution

- The problem is self-adjoint so all eigenvalues must be real.
- One can check as in Problem 1 that if $\lambda < 0$, then the only solution is $u = 0$ so $\lambda$ is not an eigenvalue.
- If $\lambda = 0$, then $u = \text{constant}$ is a solution, so $\lambda_0 = 0$ is an eigenvalue with eigenfunction $u_0 = 1$.
- If $\lambda = k^2 > 0$, then $u(x) = \cos kx$ is a solution of the ODE with $u'(0) = 0$. This function satisfies the BC $u'(1) = 0$ if $\sin k = 0$ or $k = n\pi$ for $n \in \mathbb{N}$.
- The eigenvalues are

$$\lambda_n = n^2\pi^2 \quad \text{for} \quad n = 0, 1, 2, \ldots$$

with eigenfunctions

$$u_n(x) = \cos(n\pi x) \quad \text{for} \quad n = 0, 1, 2, \ldots$$
3. Consider a wave equation

\[ u_{tt} - \left( c_0^2 u_x \right)_x + q_0 u = 0 \]

for \( u(x,t) \) where the wave speed \( c_0(x) \) and the coefficient \( q_0(x) \) depend on \( x \). Look for (possibly complex-valued) separable solutions of the form

\[ u(x,t) = X(x) T(t). \]

Show that, up to linear superpositions, we can take

\[ T(t) = e^{-i\omega t} \]

for some constant \( \omega \in \mathbb{C} \). Find the corresponding ODE satisfied by \( X \) in that case. (Note that if a function of \( x \) is equal to a function of \( t \), then the functions must be constant.)

**Solution**

- Using \( u(x,t) = X(x) T(t) \) in the wave equation, we get

\[ XT'' - \left( c_0^2 X' \right)' T + q_0 XT = 0. \]

Dividing this equation by \( XT \) and rearranging the result, we get that

\[ \frac{T''}{T} = \frac{(c_0^2 X')'}{X} - q_0. \]

- The left-hand side is function of \( t \) only and the right-hand side is a function of \( x \) only, so each function must be constant, equal to \( \lambda = -\omega^2 \), say, for some \( \omega \in \mathbb{C} \).

- It follows that

\[ T'' + \omega^2 T = 0 \]

so (if \( \omega \neq 0 \))

\[ T(t) = c_1 e^{-i\omega t} + c_2 e^{i\omega t}. \]

- The corresponding Sturm-Liouville ODE for \( X(x) \) is

\[- \left( c_0^2 X' \right)' + q_0 X = \omega^2 X. \]
4. Define the Hermite polynomials $H_n$ by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \left(e^{-x^2}\right)$$

and let

$$\phi_n(x) = e^{-x^2/2}H_n(x)$$

where $n = 0, 1, 2, 3, \ldots$. Show that $\phi_n$ is a solution of the Sturm-Liouville problem

$$-\phi''_n + x^2 \phi_n = \lambda_n \phi_n \quad -\infty < x < \infty,$$

$$\phi_n(x) \to 0 \quad \text{as} \quad |x| \to \infty$$

with eigenvalue

$$\lambda_n = 2n + 1.$$

**HINT:** Let $L$ be the linear operator

$$L = -\frac{d^2}{dx^2} + x^2,$$

meaning that $L$ acts on functions $\phi$ by

$$L\phi = -\phi'' + x^2\phi.$$

Define operators $A$, $A^*$ by

$$A = \frac{d}{dx} + x, \quad A^* = -\frac{d}{dx} + x.$$

Show that

$$A\phi_n = 2n\phi_{n-1}, \quad A^* \phi_n = \phi_{n+1} \quad (1)$$

and

$$L = AA^* - 1.$$

**Solution**

- Note that

$$(xu)' = xu' + u,$$

which means that

$$\frac{d}{dx} x = x \frac{d}{dx} + 1,$$
or
\[
\left[ \frac{d}{dx}, x \right] = \frac{d}{dx} x - x \frac{d}{dx} = 1.
\]
Hence
\[
AA^* = \left( \frac{d}{dx} + x \right) \left( -\frac{d}{dx} + x \right)
= -\frac{d^2}{dx^2} - x \frac{d}{dx} + \frac{d}{dx} x + x^2
= -\frac{d^2}{dx^2} + x^2 + 1
= L + 1.
\]

- Assuming (1), we get that
\[
L\phi_n = (AA^* - 1) \phi_n
= AA^*\phi_n - \phi_n
= A\phi_{n+1} - \phi_n
= 2(n + 1)\phi_n - \phi_n
= (2n + 1)\phi_n,
\]
which shows that $\phi_n$ is an eigenfunction of $L$ with eigenvalue $2n + 1$.

- We have
\[
\phi_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} \left( e^{-x^2} \right).
\]
Hence
\[
A^*\phi_n = (-1)^n \left( -\frac{d}{dx} + x \right) \left[ e^{x^2/2} \frac{d^n}{dx^n} \left( e^{-x^2} \right) \right]
= (-1)^{n+1} \frac{d}{dx} \left[ e^{x^2/2} \frac{d^n}{dx^n} \left( e^{-x^2} \right) \right]
+ (1)^n x e^{x^2/2} \frac{d^n}{dx^n} \left( e^{-x^2} \right)
= (-1)^{n+1} e^{x^2/2} \frac{d^{n+1}}{dx^{n+1}} \left( e^{-x^2} \right)
= \phi_{n+1}.
We have

\[ A\phi_n = (-1)^n \left( \frac{d}{dx} + x \right) \left[ e^{x^2/2} \frac{d^n}{dx^n} (e^{-x^2}) \right] \]

\[ = (-1)^n e^{x^2/2} \left[ \frac{d^{n+1}}{dx^{n+1}} (e^{-x^2}) + 2x \frac{d^n}{dx^n} (e^{-x^2}) \right]. \]

Since derivatives of \( x \) of the order greater than or equal to two vanish, the Leibnitz formula

\[ \frac{d^n}{dx^n} (fg) = f \frac{d^n g}{dx^n} + n \frac{df}{dx} \frac{d^{n-1} g}{dx^{n-1}} + \frac{1}{2} n(n-1) \frac{d^2 f}{dx^2} \frac{d^{n-2} g}{dx^{n-2}} + \cdots + \frac{d^n f}{dx^n} g \]

implies that

\[ \frac{d^{n+1}}{dx^{n+1}} (e^{-x^2}) = -2 \frac{d^n}{dx^n} (xe^{-x^2}) \]

\[ = -2 \left[ x \frac{d^n}{dx^n} (e^{-x^2}) + n \frac{d^{n-1}}{dx^{n-1}} (e^{-x^2}) \right]. \]

It follows that

\[ A\phi_n = 2n(-1)^{n-1} e^{x^2/2} \frac{d^{n-1}}{dx^{n-1}} (e^{-x^2}) \]

\[ = 2n\phi_{n-1} \]

which completes the proof of (1).