1. Suppose that $u(x)$ is a solution of the Sturm-Liouville problem with non-homogeneous ODE and BCs

$$- (pu')' + qu = f(x) \quad a < x < b,$$
$$u(a) = A, \quad u(b) = B.$$ 

Write

$$u(x) = A \left( \frac{b-x}{b-a} \right) + B \left( \frac{x-a}{b-a} \right) + v(x)$$

and show that $v$ satisfies a Sturm-Liouville problem of the form

$$- (pv')' + qv = g(x) \quad a < x < b,$$
$$v(a) = 0, \quad v(b) = 0$$

with homogeneous BCs.

2. Consider the nonhomogeneous Sturm-Liouville problem

$$- (pu')' + qu = \lambda u + f(x) \quad a < x < b,$$
$$u(a) = 0, \quad u(b) = 0.$$ 

If $\lambda$ is an eigenvalue with eigenfunction $\phi$, show that the problem only has a solution if $f$ satisfies

$$\int_a^b f\phi \, dx = 0.$$ 

Under what conditions on $f$ is the BVP

$$- u'' = f(x) \quad 0 < x < 1,$$
$$u'(0) = 0, \quad u'(1) = 0$$

solvable? How about the BVP

$$- u'' = f(x) \quad 0 < x < 1,$$
$$u'(0) = 0, \quad u'(1) = 1.$$
3. Consider the weighted Sturm-Liouville eigenvalue problem

\[-(pu')' + qu = \lambda ru \quad a < x < b,\]
\[u(a) = 0, \quad u(b) = 0\]

where \(p(x), q(x), r(x)\) are given real-valued coefficient functions and \(r > 0\).

Let \(L^2_r(a, b)\) denote the space of functions \(f : [a, b] \to \mathbb{C}\) such that

\[\int_a^b r|f|^2 \, dx < \infty\]

with weighted inner product

\[(f, g)_r = \int_a^b rf \overline{g} \, dx.\]

(a) If \(\phi(x)\) is an eigenfunction with eigenvalue \(\lambda \in \mathbb{C}\), show that \(\lambda \in \mathbb{R}\) is real.

(b) If \(\phi(x), \psi(x)\) are eigenfunctions with distinct eigenvalues \(\lambda, \mu\) show that they are orthogonal with respect to the weighted inner-product, meaning that

\[\int_a^b r\phi \overline{\psi} \, dx = 0.\]

(c) Suppose that the eigenvalue problem has a complete set of eigenfunctions \(\{\phi_n : n = 1, 2, 3, \ldots\}\). If \(f \in L^2_r(a, b)\), give an expression for the coefficients \(c_n\) in the eigenfunction expansion

\[f(x) = \sum_{n=1}^\infty c_n \phi_n(x).\]

4. Use separation of variables to solve the following IBVP for \(u(x, t)\) for the wave equation:

\[u_{tt} = u_{xx} \quad 0 < x < 1,\]
\[u_x(0, t) = 0, \quad u(1, t) = 0,\]
\[u(x, 0) = f(x), \quad u_t(x, 0) = g(x).\]