1. (a) Explain why the total birth rate $B(t)$ of a population with constant reproductive rate $\lambda$ per individual and exponential survival rate $e^{-\beta t}$ over time $t$ satisfies the renewal equation

$$B(t) = N_0 \lambda e^{-\beta t} + \lambda \int_0^t e^{-\beta(t-s)} B(s) \, ds$$

where $N_0$ is the total initial population at $t = 0$. What are the dimensions of $N_0$, $\lambda$, $\beta$, and $B(t)$?

(b) Solve this integral equation and discuss the behavior of $B(t)$ as $t \to \infty$. Does your answer make sense?

**HINT.** One approach is to show that $B(t)$ satisfies the IVP

$$\dot{B} = (\lambda - \beta) B, \quad B(0) = N_0 \lambda.$$

**Solution**

- (a) After time $t$, there are $N_0 e^{-\beta t}$ survivors from the initial population, and they contribute $\lambda \cdot N_0 e^{-\beta t}$ to the birthrate. For $0 < s < t$, the population born between times $s$ and $s + ds$ is $B(s) \, ds$, and of these $e^{\beta(t-s)} B(s) \, ds$ survive to time $t$, so they contribute $\lambda e^{-\beta(t-s)} B(s) \, ds$ to the birth rate. Integrating this over $0 < s < t$ and adding the contribution form the initial population, we get the total birth rate $B(t)$.

- Let $P$ denote the dimension of population and $T$ the dimension of time. Then

$$[N_0] = P, \quad [\lambda] = \frac{1}{T}, \quad [\beta] = \frac{1}{T}, \quad [B] = \frac{P}{T}.$$

- (b) Differentiating the integral equation with respect to $t$, we get

$$\dot{B}(t) = -N_0 \lambda \beta e^{-\beta t} - \lambda \beta \int_0^t e^{-\beta(t-s)} B(s) \, ds + \lambda B(t).$$

Using the integral equation to eliminate the integral from this equation, we find that

$$\dot{B} = (\lambda - \beta) B.$$
Setting $t = 0$ in the integral equation, we get

$$B(0) = N_0 \lambda.$$ 

The solution of this IVP is

$$B(t) = N_0 \lambda e^{(\lambda - \beta)t}.$$

- If $\lambda > \beta$, meaning that the birth rate per individual exceeds the death rate, then the total birth rate (and the total population) grows exponentially in time. If $\lambda < \beta$, meaning that the death rate per individual exceeds the birth rate, then the total birth rate (and the total population) decays exponentially. If $\lambda = \beta$, then the total birth rate (and the total population) remains constant.
2. (a) Consider the following Fredholm equation of the second kind

\[ u(x) - \lambda \int_0^1 x y u(y) \, dy = f(x), \quad 0 \leq x \leq 1 \]

where \( \lambda \in \mathbb{C} \) is a constant and \( f \) is a given (continuous) function. If \( \lambda \neq 3 \), show that this equation has a unique (continuous) solution for \( u(x) \) and find the solution. If \( \lambda = 3 \), determine for what functions \( f \) a solution exists and find the solutions in that case.

(b) For what functions \( f \) is the following Fredholm equation of the first kind

\[ \int_0^1 x y u(y) \, dy = f(x), \quad 0 \leq x \leq 1 \]

solvable for \( u(x) \)? Describe the solutions in that case.

HINT. Note that these Fredholm equations are degenerate.

Solution

- (a) The integral equation implies that

\[ u(x) = f(x) + cx \quad (1) \]

where the constant \( c \) is given by

\[ c = \lambda \int_0^1 y u(y) \, dy. \]

Using the expression (1) for \( u \) in this equation for \( c \), we get

\[ c = \lambda \int_0^1 y f(y) \, dy + \lambda \int_0^1 y \cdot cy \, dy \]

\[ = \lambda \int_0^1 y f(y) \, dy + \frac{1}{3} \lambda c, \]

or

\[ (3 - \lambda) c = 3 \lambda \int_0^1 y f(y) \, dy. \]
• If \( \lambda \neq 3 \), then we get
\[
c = \frac{3\lambda}{3 - \lambda} \int_0^1 yf(y) \, dy
\]
and (1) with this value of \( c \) is the unique solution of the integral equation for any continuous function \( f \).

• If \( \lambda = 3 \), then the integral equation is solvable if and only if
\[
\int_0^1 yf(y) \, dy = 0.
\]
In that case, the solution is (1) where \( c \) is an arbitrary constant.

• Note that the integral operator
\[
(Ku)(x) = \int_0^1 xyu(y) \, dy
\]
has eigenvalue \( \mu = 1/3 \) and eigenfunction \( u(x) = x \) with \( Ku = \mu u \), so \( cx \) is an arbitrary solution of the homogeneous equation for \( \lambda = 3 \).

• (c) Since
\[
\int_0^1 xyu(y) \, dy = cx, \quad c = \int_0^1 yu(y) \, dy
\]
the equation is only solvable if
\[
f(x) = cx
\]
for some constant \( c \). In that case, one solution is the constant function
\[
u(x) = 2c
\]
The general solution is
\[
u(x) = 2c + v(x)
\]
where \( v(x) \) is any function orthogonal to \( x \), meaning that
\[
\int_0^1 xv(x) \, dx = 0.
\]

• Note that the range of \( K \) is one-dimensional, and the nullspace of \( K \), consisting of eigenfunctions with eigenvalue \( \mu = 0 \), is infinite-dimensional.
3. Show that the following IVP for a second-order scalar ODE for \( u(t) \)

\[
\ddot{u}(t) = f(t, u(t)),
\]

\[
 u(0) = u_0, \quad \dot{u}(0) = v_0
\]
is equivalent to the Volterra integral equation

\[
u(t) = \int_0^t (t - s) f(s, u(s)) \, ds + u_0 + v_0 t.\]

**Solution**

- Integrating the ODE once, we get

\[
\dot{u}(t) = v_0 + \int_0^t f(s, u(s)) \, ds.
\]

Integrating again, we get

\[
u(t) = u_0 + v_0 t + \int_0^t \left[ \int_0^r f(s, u(s)) \, ds \right] \, dr.
\]

Integrating by parts in the \( r \)-integral, we obtain

\[
\int_0^t \left[ \int_0^r f(s, u(s)) \, ds \right] \, dr = \int_0^t 1 \cdot \left[ \int_0^r f(s, u(s)) \, ds \right] \, dr
\]

\[
= \left[ r \int_0^r f(s, u(s)) \, ds \right]_0^t - \int_0^t r f(r, u(r)) \, dr
\]

\[
= t \int_0^t f(s, u(s)) \, ds - \int_0^t s f(s, u(s)) \, ds
\]

\[
= \int_0^t (t - s) f(s, u(s)) \, ds,
\]

which shows that a solution of the IVP satisfies the integral equation.

- Conversely, if \( u(t) \) satisfies the integral equation, then setting \( t = 0 \) in the equation we get \( u(0) = u_0 \). Similarly, differentiating the integral equation once and setting \( t = 0 \), we get \( \dot{u}(0) = v_0 \). Differentiating once again, we get \( \ddot{u} = f(t, u) \), so a solution of the integral equation satisfies the IVP.

- Note that the integral equation incorporates both the ODE and the initial conditions.
4. Consider the following BVP for \( u(x) \) in \( 0 < x < 1 \):

\[
-u'' = k^2 [1 + \epsilon q(x)] u + f(x),
\]

\[
u(0) = 0, \quad u(1) = 0.
\]

Here \( k > 0 \) is a constant, \( \epsilon \) is a small parameter, and \( f(x), q(x) \) are given (continuous) functions. Assume that \( k \neq n\pi \) for any integer \( n \in \mathbb{N} \), so that \( k^2 \) is not an eigenvalue of \(-d^2/dx^2\) with Dirichlet BCs.

(a) Find the Green’s function \( G(x, \xi) \) for (2) with \( \epsilon = 0 \), which satisfies

\[
-\frac{d^2 G}{d^2 x} = k^2 G + \delta(x - \xi) \quad \text{in } 0 < x < 1,
\]

\[
G(0, \xi) = 0, \quad G(1, \xi) = 0.
\]

(b) Use the Green’s function from (a) to reformulate (2) as a Fredholm integral equation for \( u(x) \) of the form

\[
u(x) = \epsilon \int_0^1 K(x, \xi) u(\xi) \, d\xi + g(x).
\]

(c) Write out the first few terms in the Neumann series (or Born approximation) for \( u \) as integrals involving \( g(x), q(x), \) and \( G(x, \xi) \).

Solution

- (a) Solutions of the homogeneous equation

\[
-\frac{d^2 u}{dx^2} - k^2 u = 0
\]

that vanish at \( x = 0 \) and \( x = 1 \) are

\[
u_1(x) = \sin k x, \quad u_2(x) = \sin (k(1 - x))
\]

respectively. The Wronskian of these solutions is

\[
u_1 u_2' - u_1 u_2' = -k \sin k x \cos (k(1 - x)) - k \cos k x \sin (k(1 - x))
\]

\[
= -k \sin k,
\]

which is nonzero if \( k \neq n\pi \).
The Green’s function in (3), whose $x$-derivative jumps by $-1$ across $x = \xi$, is therefore
\[
G(x, \xi) = \begin{cases} 
\sin kx \sin (k(1 - \xi)) / k \sin k & \text{if } 0 \leq x \leq \xi, \\
\sin k \xi \sin (k(1 - x)) / k \sin k & \text{if } \xi \leq x \leq 1.
\end{cases}
\]
or
\[
G(x, \xi) = \frac{\sin (kx<) \sin (k(1 - x>))}{k \sin k}.
\]

(b) We write (2) as
\[
- u'' - k^2 u = \epsilon q(x) u + f(x),
\]
\[
u(0) = 0, \quad u(1) = 0.
\]

It follows from the Green’s function representation that
\[
u(x) = \int_0^1 G(x, \xi) [\epsilon q(\xi) u(\xi) + f(\xi)] d\xi
= \epsilon \int_0^1 G(x, \xi) q(\xi) u(\xi) d\xi + \int_0^1 G(x, \xi) f(\xi) d\xi,
\]
which gives (4) with
\[
K(x, \xi) = G(x, \xi) q(\xi), \quad g(x) = \int_0^1 G(x, \xi) f(\xi) d\xi.
\]

(c) The Neumann series expansion for
\[
u = g + \epsilon Ku
\]
is
\[
u = g + \epsilon Kg + \epsilon^2 K^2 g + \epsilon^3 K^3 g + \ldots
\]

Explicitly, we have
\[
Kg(x) = \int_0^1 G(x, \xi) q(\xi) g(\xi) d\xi;
\]
\[
K^2 g(x) = \int_0^1 \int_0^1 G(x, \xi_1) G(x, \xi_2) q(\xi_1) q(\xi_2) g(\xi_2) d\xi_1 d\xi_2;
\]
\[
K^2 g(x) = \int_0^1 \int_0^1 \int_0^1 G(x, \xi_1) G(x, \xi_2) G(x, \xi_3) q(\xi_1) q(\xi_2) q(\xi_3) g(\xi_3) d\xi_1 d\xi_2 d\xi_3.
\]

These terms correspond physically to single, double, and triple scattering corrections due to the perturbation $\epsilon q(x)$ in the coefficient.