1. Let \( p, q, r : [a, b] \to \mathbb{R} \) be smooth functions with \( p, r > 0 \). Suppose that \( u : [a, b] \to \mathbb{R} \) is a smooth extremal over all functions in \( C^1([a, b]) \) of the functional
\[
J(u) = \frac{1}{2} \int_a^b \{ p(x)u'^2(x) + q(x)u^2(x) \} \, dx
\]
subject to the constraint \( K(u) = 1 \), where
\[
K(u) = \frac{1}{2} \int_a^b r(x)u^2(x) \, dx.
\]
Show that \( u \) is an eigenfunction of the weighted Sturm-Liouville eigenvalue problem
\[
-(pu')' + qu = \lambda ru \quad a < x < b,
\]
\[
ue(a) = 0, \quad u'e(b) = 0.
\]

Solution.

• Introduce a Lagrange multiplier \( \lambda \). Then \( u \) is an unconstrained extremal of
\[
I(u, \lambda) = J(u) - \lambda K(u),
\]
and (provided that \( \delta K/\delta u \neq 0 \), which is the case since \( ru \neq 0 \))
\[
\frac{d}{d\epsilon} J(u + \epsilon h) \bigg|_{\epsilon=0} - \lambda \frac{d}{d\epsilon} K(u + \epsilon h) \bigg|_{\epsilon=0} = 0 \quad \text{for every } h \in C^\infty([a, b]).
\]

• Using integration by parts, we get that
\[
\frac{d}{d\epsilon} J(u + \epsilon h) \bigg|_{\epsilon=0} = \int_a^b \{ pu'h' + quh \} \, dx = \left[ pu'h \right]_a^b + \int_a^b \{ -(pu')' + qu \} \, h \, dx,
\]
\[
\frac{d}{d\epsilon} K(u + \epsilon h) \bigg|_{\epsilon=0} = \int_a^b ruh \, dx.
\]
• It follows that

\[ [pu'h]_a^b + \int_a^b \{-(pu')' + qu - \lambda ru \} \, h \, dx = 0 \quad \text{for every } h \in C^\infty([a,b]). \]

• First, considering compactly supported \( h \) with \( h(a) = 0, h(b) = 0 \), we get that

\[ \int_a^b \{-(pu')' + qu - \lambda ru \} \, h \, dx = 0 \quad \text{for every } h \in C^\infty_c([a,b]). \]

Since the integrand is a continuous function, the fundamental lemma of the calculus of variations implies that

\[-(pu')' + qu - \lambda ru = 0 \quad a < x < b.\]

• It then follows that

\[ [pu'h]_a^b = 0 \quad \text{for every } h \in C^\infty([a,b]). \]

Choosing \( h \) such that \( h(a) = 1, h(b) = 0 \) or \( h(a) = 0, h(b) = 1 \) we find that \( u \) satisfies the natural boundary conditions

\[ u'(a) = 0, \quad u'(b) = 0. \]
2. Solve the following IBVP for \( u(x, t) \) by the method of separation of variables:

\[
\begin{align*}
    u_{tt} - u_{xx} + u &= 0, \quad 0 < x < 1, \quad t > 0 \\
    u_x(0, t) &= 0, \quad u_x(1, t) = 0, \\
    u(x, 0) &= f(x), \quad u_t(x, 0) = 0.
\end{align*}
\]

Solution.

- Looking for separated solutions \( u(x, t) = F(x)G(t) \), we get that

\[
\frac{G''}{G} - \frac{F''}{F} + 1 = 0.
\]

Introducing a separation constant \( \lambda = -\frac{F''}{F} \), we find that \( F \) satisfies the eigenvalue problem

\[
-F'' = \lambda F, \quad F'(0) = 0, \quad F'(1) = 0,
\]

and \( G \) satisfies the ODE

\[
G'' + (\lambda + 1)G = 0.
\]

- The eigenvalues and eigenfunctions are

\[
\lambda = n^2 \pi^2, \quad F(x) = \cos n\pi x \quad n = 0, 1, 2, \ldots
\]

- The corresponding solution for \( G \) is

\[
G(t) = A \cos \omega_n t + B \sin \omega_n t \quad \omega_n = \sqrt{n^2 \pi^2 + 1}.
\]

Since \( u_t = 0 \) at \( t = 0 \), we take \( B = 0 \).

- Superposing separated solutions, we get that

\[
u(x, t) = \sum_{n=0}^{\infty} a_n \cos(\omega_n t) \cos(n\pi x).
\]

where the coefficients \( a_n \) are chosen so that

\[
f(x) = \sum_{n=0}^{\infty} a_n \cos(n\pi x).
\]

- By orthogonality, it follows that

\[
a_0 = \int_0^1 f(x) \, dx, \quad a_n = 2 \int_0^1 f(x) \cos(n\pi x) \, dx \quad n \geq 1.
\]
3. Suppose that $f : [0, 1] \rightarrow \mathbb{R}$ is a continuous function and the boundary-value problem

\[-u'' = \pi^2 u + f(x), \quad u(0) = 0, \quad u(1) = 0\]

has a solution $u \in C^2([0, 1])$. Show that $f$ must satisfy the solvability condition

$$\int_0^1 f(x) \sin(\pi x) \, dx = 0.$$

**Solution.**

- Multiplying the ODE by $\sin(\pi x)$, integrating over $[0, 1]$, and integrating by parts, we get that

$$\int_0^1 f(x) \sin(\pi x) \, dx = - \int_0^1 (u'' + \pi^2 u) \sin(\pi x) \, dx$$

$$= [\pi u \cos(\pi x) - u' \sin(\pi x)]_0^1 - \int_0^1 u \left\{[\sin(\pi x)]'' + \pi^2 \sin(\pi x)\right\} \, dx$$

$$= 0.$$

The boundary terms vanish since both $u$ and $\sin(\pi x)$ are zero at $x = 0, 1$.

**Remark.** In general, if $A$ is a self-adjoint operator on a Hilbert space with eigenvalue $\lambda \in \mathbb{R}$ and eigenfunction $\phi$, then a necessary condition for the equation $Au = \lambda u + f$ to have a solution for $u$ is that $f$ is orthogonal to $\phi$, since

$$\langle f, \phi \rangle = \langle (A - \lambda I)u, \phi \rangle = \langle u, (A - \lambda I)\phi \rangle = 0.$$
4. Let \( A : D(A) \subset L^2(0, 2\pi) \to L^2(0, 2\pi) \) be the operator

\[
A = -i \frac{d}{dx} + x,
\]

with periodic boundary conditions

\[
D(A) = \{ u : [0, 2\pi] \to \mathbb{C} : u \in H^1(0, 2\pi), u(0) = u(2\pi) \}
\]

(a) Show that \( A \) is self-adjoint.

(b) Compute the eigenvalues and eigenfunctions of \( A \).

Solution.

• (a) For \( u, v \in D(A) \), we have

\[
\langle u, Av \rangle = \int_0^{2\pi} \overline{u}(-iv' + xv) \, dx
\]

\[
= -i \overline{u}v|_0^{2\pi} + \int_0^{2\pi} \{i\overline{u}' + xu\} v \, dx
\]

\[
= \int_0^{2\pi} \{-i\overline{u}' + xu\} v \, dx
\]

\[
= \langle Au, v \rangle
\]

The boundary terms vanish since both \( u \) and \( v \) satisfy periodic boundary conditions.

• (b) The eigenvalue problem for \( A \) is

\[
-iu' + xu = \lambda u, \quad u(0) = u(2\pi).
\]

Solving the ODE by separating variables, we get that

\[
-i \int \frac{du}{u} = \int (\lambda - x) \, dx
\]

so \( \log u = i(\lambda x - x^2/2) + C \), and a nonzero solution is

\[
u(x) = e^{ix(\lambda - x/2)}.
\]
• This function satisfies the periodic boundary conditions if

\[ 1 = e^{2\pi i (\lambda - \pi)}, \]

which means that \( \lambda - \pi = n \) for some \( n \in \mathbb{Z} \).

• It follows that the eigenvalues \( \lambda = \lambda_n \) and eigenfunctions \( u = u_n \) are given by

\[ \lambda_n = n + \pi, \quad u_n(x) = e^{i(nx + \pi x - x^2/2)} \quad n \in \mathbb{Z}. \]