1. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(0,0) = 0$ and

$$f(x, y) = \frac{xy^3}{x^2 + y^6} \quad \text{if } (x, y) \neq (0,0).$$

(a) Show that the directional derivatives of $f$ at $(0,0)$ exist in every direction. What is its Gâteaux derivative at $(0,0)$?

(b) Show that $f$ is not Fréchet differentiable at $(0,0)$. (HINT. A Fréchet differentiable function must be continuous.)

2. Define $f, g : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = x^2 + y^2, \quad g(x, y) = (y - 1)^3 - x^2.$$ 

Find the minimum value of $f(x, y)$ subject to the constraint $g(x, y) = 0$. Show that there does not exist any constant $\lambda$ such that $\nabla f = \lambda \nabla g$ at some point $(x, y) \in \mathbb{R}^2$. Why does the method of Lagrange multipliers fail in this example?

3. Derive the Euler-Lagrange equation for a functional of the form

$$J(u) = \int_a^b F(x, u, u', u'') \, dx.$$ 

What are the natural boundary conditions for this functional?

4. A curve $y = u(x)$ with $a \leq x \leq b$, $u(x) > 0$, and $u(a) = u_0$, $u(b) = u_1$ is rotated about the $x$-axis. Find the curve that minimizes the area of the surface of revolution,

$$J(u) = \int_a^b u \sqrt{1 + (u')^2} \, dx.$$
5. Let \( X \) be the space of smooth functions \( u : [0, 1] \to \mathbb{R} \) such that \( u(0) = 0 \), \( u(1) = 0 \). Define functionals \( J, K : X \to \mathbb{R} \) by

\[
J(u) = \frac{1}{2} \int_0^1 (u')^2 \, dx, \quad K(u) = \frac{1}{2} \int_0^1 u^2 \, dx.
\]

(a) Introduce a Lagrange multiplier and write down the Euler-Lagrange equation for extremals in \( X \) of the functional \( J(u) \) subject to the constraint \( K(u) = 1 \).
(b) Solve the eigenvalue problem in (a) and find all of the extremals. Which one minimizes \( J(u) \)?

6. (a) Make a change of variable \( x = \phi(t), \ v(t) = u(\phi(t)) \), where \( \phi'(t) > 0 \), in the functional

\[
J(u) = \int_a^b F(x, u, u') \, dx.
\]

Show that \( J(u) = K(v) \) where \( K(v) \) has the form

\[
K(v) = \int_c^d G(t, v, v') \, dt
\]

and express \( G \) in terms of \( F \) and \( \phi \).
(b) Show that the Euler-Lagrange equation for \( K(v) \) is the same as what you get by changing variables in the Euler-Lagrange equation for \( J(u) \).