Problem set 2: Solutions

1. A particle of mass $m$ with position $\vec{x}(t)$ at time $t$ has potential energy $V(\vec{x})$ and kinetic energy

$$T = \frac{1}{2} m |\vec{x}_t|^2.$$ 

The action of the particle over times $t_0 \leq t \leq t_1$ is the time-integral of the difference between the kinetic and potential energy:

$$S(\vec{x}) = \int_{t_0}^{t_1} (T - V) \, dt.$$ 

(a) Show that an extremal $\vec{x}(t)$ of $S$ satisfies Newton’s second law $\vec{F} = m \vec{a}$ for motion in a conservative force field $\vec{F} = -\nabla V$.

(b) Show that the total energy of the particle $E = T + V$ is a constant independent of time.

Solution.

- (a) The Euler-Lagrange equation for the action

$$S(\vec{x}) = \int_{t_0}^{t_1} L(\vec{x}, \vec{x}_t) \, dt, \quad L(\vec{x}, \vec{x}_t) = \frac{1}{2} m |\vec{x}_t|^2 - V(\vec{x})$$ 

is given by

$$-\frac{d}{dt} \left( \frac{\partial L}{\partial \vec{x}_t} \right) + \frac{\partial L}{\partial \vec{x}} = 0.$$ 

- Using the summation convention, and the Kronecker-$\delta$ defined by

$$\delta_{ij} = \begin{cases} 
1 & \text{if } i = j, \\
0 & \text{if } i \neq j, 
\end{cases}$$

we have

$$\left[ \frac{\partial |\vec{a}|^2}{\partial \vec{a}} \right]_i = \frac{\partial}{\partial a_i} (a_j a_j) = 2 a_j \delta_{ij} = 2 a_i = [2\vec{a}]_i,$$ 

so $\partial F/\partial \vec{x}_t = m \vec{x}_t$, and $\partial F/\partial \vec{x} = -\nabla V(\vec{x})$. 

• The Euler-Lagrange equation is

\[ m\ddot{x} = -\nabla V(x), \]

which is Newton’s second law.

• (b) Taking the scalar product of the ODE with \( \dot{x} \), we get that

\[ m\dot{x} \cdot \ddot{x} = -\dot{x} \cdot \nabla V(x). \]

It follows that

\[ \frac{d}{dt} \left[ \frac{1}{2} m\dot{x} \cdot \dot{x} + V(x) \right] = 0, \]

so

\[ \frac{1}{2} m|\dot{x}|^2 + V(x) = \text{constant}. \]
2. Let $\Omega \subset \mathbb{R}^n$ be a bounded region with smooth boundary (so the divergence theorem holds) and $f : \overline{\Omega} \to \mathbb{R}$ a smooth function. Derive the Euler-Lagrange equation and natural boundary condition that are satisfied by a smooth extremal $u : \overline{\Omega} \to \mathbb{R}$ of the functional

$$J(u) = \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 - fu \right) \, dx.$$ 

**Solution.**

- For $\phi \in C^\infty(\overline{\Omega})$, we have

$$\frac{d}{d\epsilon} J(u + \epsilon \phi)|_{\epsilon=0} = \frac{d}{d\epsilon} \int_{\Omega} \left[ \frac{1}{2} (\nabla u + \epsilon \nabla \phi) \cdot (\nabla u + \epsilon \nabla \phi) - f(u + \epsilon \phi) \right] \, dx \bigg|_{\epsilon=0} = \int_{\Omega} (\nabla u \cdot \nabla \phi - f \phi) \, dx.$$

- The divergence theorem implies that

$$\int_{\Omega} (\nabla u \cdot \nabla \phi) \, dx = \int_{\Omega} [\nabla \cdot (\phi \nabla u) - \phi \Delta u] \, dx = \int_{\partial \Omega} \phi \frac{\partial u}{\partial n} \, dS - \int_{\Omega} \phi \Delta u \, dx.$$

- If $\phi = 0$ on $\partial \Omega$, then

$$\frac{d}{d\epsilon} J(u + \epsilon \phi)|_{\epsilon=0} = - \int_{\Omega} (\Delta u + f) \phi \, dx.$$

The fundamental lemma of the calculus of variations implies that an extremal $u$, with

$$\frac{d}{d\epsilon} J(u + \epsilon \phi)|_{\epsilon=0} = 0,$$

for all $\phi$ satisfies the Euler-Lagrange equation

$$-\Delta u = f \quad \text{in} \ \Omega.$$
For functions \( \phi \) that do not necessarily vanish on the boundary, we have
\[
\frac{d}{d\epsilon} J(u + \epsilon \phi)|_{\epsilon=0} = \int_{\partial \Omega} \phi \frac{\partial u}{\partial n} \, dS.
\]
It follows that the natural boundary condition for an extremal \( u \) is the Neumann condition
\[
\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \Omega.
\]
3. The transverse displacement at position \( x \) and time \( t \) of an elastic string vibrating in the \((x, y)\)-plane is given by \( y = u(x, t) \), where \( a \leq x \leq b \) and \( t_0 \leq t \leq t_1 \). If the density of the string per unit length is \( \rho(x) \) and the tension in the string is a constant \( T \), then (for small displacements) the motion of the string is an extremum of the action

\[
S(u) = \int_{t_0}^{t_1} \int_a^b \left( \frac{1}{2} \rho u_t^2 - \frac{1}{2} T u_x^2 \right) \, dx \, dt.
\]

Derive the Euler-Lagrange equation for \( u(x, t) \).

Solution.

- For every \( \phi \in C_0^\infty ((a, b) \times (t_0, t_1)) \), we have

\[
\frac{d}{d \epsilon} S(u + \epsilon \phi) \big|_{\epsilon = 0} = \int_{t_0}^{t_1} \int_a^b (\rho u_t \phi_t - T u_x \phi_x) \, dx \, dt.
\]

- Integrating by parts and using the fact that \( \phi = 0 \) at \( x = a, b \) and \( t = t_0, t_1 \), we get that

\[
\frac{d}{d \epsilon} S(u + \epsilon \phi) \big|_{\epsilon = 0} = \int_{t_0}^{t_1} \int_a^b (-\rho u_{tt} + T u_{xx}) \phi \, dx \, dt.
\]

- The fundamental lemma of the calculus of variations then implies that a smooth extremal \( u \) of \( S \) satisfies the wave equation

\[
\rho u_{tt} = T u_{xx}.
\]
4. The \((n\text{-dimensional})\) area of a surface \(y = u(x)\) over a region \(\Omega \subset \mathbb{R}^n\) is given by

\[ J(u) = \int_{\Omega} \sqrt{1 + |\nabla u|^2} \, dx. \]

Find the Euler-Lagrange equation (called the minimal surface equation) that is satisfied by a smooth extremum of this functional.

Solution.

- Using the divergence theorem, we find for every \(\phi \in C_c^\infty(\Omega)\) that

\[
\frac{d}{d\epsilon} J(u + \epsilon \phi)\big|_{\epsilon=0} = \frac{d}{d\epsilon} \int_{\Omega} \sqrt{1 + |\nabla u + \epsilon \nabla \phi|^2} \, dx\big|_{\epsilon=0} \\
= \int_{\Omega} \frac{1}{2\sqrt{1 + |\nabla u|^2}} \frac{d}{d\epsilon} |\nabla u + \epsilon \nabla \phi|^2 \big|_{\epsilon=0} \, dx \\
= \int_{\Omega} \frac{\nabla u \cdot \nabla \phi}{\sqrt{1 + |\nabla u|^2}} \, dx \\
= - \int_{\Omega} \nabla \cdot \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) \phi \, dx.
\]

- The Euler-Lagrange equation for smooth extremals of \(J\) is therefore

\[
\nabla \cdot \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0
\]

- Using the summation convention, we can write this equation in component form as

\[
\left( \frac{u_{x_i}}{\sqrt{1 + u_{x_j} u_{x_j}}} \right) = 0.
\]
5. Let \( X = \{ u \in C^1([-1,1]) : u(-1) = -1, u(1) = 1 \} \), where \( C^1([a,b]) \) denotes the space of continuously differentiable functions on \([a,b]\). Define \( J : X \to \mathbb{R} \) by
\[
J(u) = \int_{-1}^{1} x^4 (u')^2 \, dx.
\]

(a) Show that 
\[
\inf_{u \in X} J(u) = 0,
\]
but \( J(u) > 0 \) for every \( u \in X \) (so \( J \) does not attain its infimum on \( X \)).

(b) What happens when you try to solve the Euler-Lagrange equation for extremals of \( J \)?

Solution.

- We have \( J(u) \geq 0 \) for every \( u \in X \), so \( \inf J(u) \geq 0 \).

- To make \( J(u) \) as small as possible, we want to make \( u' \) small except near \( x = 0 \), where \( u' \) can be large.

- In order to do this, let \( f : \mathbb{R} \to \mathbb{R} \) be any continuously differentiable, odd function such that \( f(x) \to 1 \) as \( x \to \infty \) and
\[
\int_{0}^{\infty} x^4 f'(x)^2 \, dx = C < \infty.
\]
For example, we could choose \( f(x) = \tanh x \).

- If we were to include piecewise smooth functions in \( X \), or functions in a Sobolev space such as \( H^1(-1,1) \), we could instead take
\[
f(x) = \begin{cases} 
-1 & \text{if } -\infty < x \leq -1, \\
x & \text{if } -1 < x < 1, \\
1 & \text{if } 1 \leq x < \infty.
\end{cases}
\]

- For \( \epsilon > 0 \), define
\[
u^{\epsilon}(x) = \frac{f(x/\epsilon)}{f(1/\epsilon)}.
\]
Then \( u^{\epsilon}(-1) = -1 \) and \( u^{\epsilon}(1) = 1 \) so \( u^{\epsilon} \in X \).
Making the change of variables $x = \epsilon t$, we have
\begin{align*}
J(u') &= \frac{2}{\epsilon^2 f(1/\epsilon)^2} \int_0^1 x^4 \left[ f'(\frac{x}{\epsilon}) \right]^2 \, dx \\
&= \frac{2\epsilon^3}{f(1/\epsilon)^2} \int_0^{1/\epsilon} t^4 f'(t)^2 \, dt \\
&\leq \frac{2C\epsilon^3}{f(1/\epsilon)^2}.
\end{align*}
It follows that $J(u') \to 0$ as $\epsilon \to 0^+$, so
\[
\inf_{u \in X} J(u) = 0.
\]

- If $J(u) = 0$ and $u \in C^1([-1, 1])$, then $x^4(u')^2 = 0$, so $u'(x) = 0$ except at $x = 0$, and then $u'(0) = 0$ by continuity. It follows that $u = \text{constant}$, but no constant function satisfies both boundary conditions for functions in $X$, so $J(u) > 0$ for every $u \in X$.

- (b) The Euler-Lagrange equation for $J(u)$ is
\[-2 \left( x^4 u' \right)' = 0.\]
Integrating this ODE once, we get that $x^4 u' = c$, where $c$ is a constant of integration. Integrating again, we get that
\[u(x) = c_1 + \frac{c_2}{x^3}\]
where $c_1$, $c_2$ are constants of integration.

- The boundary conditions imply that $c_1 - c_2 = -1$, $c_1 + c_2 = 1$, so $c_1 = 0$, $c_2 = 1$.

- The solution of the Euler-Lagrange equation is $u(x) = 1/x^3$, which is singular at $x = 0$, so there is no smooth solution.

**Remark.** The Lagrangian for $J(u)$ is $F(x, u') = x^4(u')^2$. When $x \neq 0$, the Lagrangian is a strictly convex function of $u'$ (with $F_{u'u'} = 2x^4 > 0$), but strict convexity is lost at $x = 0$. The nonexistence of a minimizer of $J(u)$ is associated with this loss of strict convexity.