1. Suppose that \( p : [a, b] \to \mathbb{R} \) is a continuously differentiable function such that \( p > 0 \), and \( q, r : [a, b] \to \mathbb{R} \) are continuous functions such that \( r > 0 \), \( q \geq 0 \). Define a weighted inner product on \( L^2(a, b) \) by

\[
\langle u, v \rangle_r = \int_a^b r(x)u(x)v(x) \, dx.
\]

Let \( A : D(A) \subset L^2(a, b) \to L^2(a, b) \) be the operator

\[
A = \frac{1}{r(x)} \left[ -\frac{d}{dx} p(x) \frac{d}{dx} + q(x) \right]
\]

with Dirichlet boundary conditions and domain

\[
D(A) = \{ u \in H^2(a, b) : u(a) = 0, u(b) = 0 \}.
\]

(a) Show that

\[
\langle u, Av \rangle_r = \langle Au, v \rangle_r \quad \text{for all } u, v \in D(A),
\]

meaning that \( A \) is self-adjoint with respect to \( \langle \cdot, \cdot \rangle_r \).

(b) Show that the eigenvalues \( \lambda \) of the weighted Sturm-Liouville eigenvalue problem

\[-(pu')' + qu = \lambda ru, \quad u(a) = 0, \quad u(b) = 0\]

are real and positive and eigenfunctions associated with different eigenvalues are orthogonal with respect to \( \langle \cdot, \cdot \rangle_r \).

**Solution.**

- (a) Using integration by parts and the boundary conditions satisfied by \( u, v \in D(A) \), we have

\[
\langle u, Av \rangle_r = \int_a^b \frac{1}{r} \left[ -(pu')' + qv \right] \, dx
= \left[ p(\bar{u}' - v') \right]_a^b + \int_a^b \left[ -(p\bar{u}')' + q\bar{u} \right] v \, dx
= \int_a^b \left[ -(pu')' + qu \right] v \, dx
= \langle Au, v \rangle_r.
\]
(b) The reality of the eigenvalues and the orthogonality of the eigenfunctions follows directly from the self-adjointness of \( A \). Moreover, if \( Au = \lambda u \) and \( u \in D(A) \) is normalized so that \( \|u\|_r = 1 \), then an integration by parts gives

\[
\lambda = \langle u, Au \rangle_r = \int_a^b \overline{u} [-pu' + qu] \, dx = \int_a^b \left[ p|u'|^2 + q|u|^2 \right] \, dx \geq 0.
\]

If \( \lambda = 0 \), then \( \int_a^b p|u'|^2 \, dx = 0 \), so \( u' = 0 \) and \( u = \text{constant} \). Then the boundary condition implies that \( u = 0 \), so \( \lambda = 0 \) is not an eigenvalue, and \( \lambda > 0 \).
2. A nonuniform string of length one with wave speed $c_0(x) = \sqrt{T/\rho_0(x)} > 0$ is fixed at each end, with zero initial displacement and nonzero initial velocity. The transverse displacement $y = u(x, t)$ of the string satisfies the IBVP

$$u_{tt} = c_0^2(x)u_{xx} \quad 0 < x < 1, \quad t > 0,$$

$$u(0, t) = 0, \quad u(1, t) = 0 \quad t > 0,$$

$$u(x, 0) = 0 \quad 0 < x < 1,$$

$$u_t(x, 0) = g(x) \quad 0 < x < 1,$$

Find the solution in terms of the eigenvalues $\lambda_n$ and eigenfunctions $\phi_n(x)$ of the weighted Sturm-Liouville problem

$$-c_0^2\phi_n'' = \lambda_n \phi_n, \quad \phi_n(0) = 0, \quad \phi_n(1) = 0, \quad n = 1, 2, 3, \ldots$$

Solution.

- Separation of variables gives the solutions

$$u(x, t) = \begin{cases} \phi_n(x) \cos(k_n t), \\ \phi_n(x) \sin(k_n t), \end{cases}$$

where $k_n > 0$ with $k_n^2 = \lambda_n$. From the previous question, $\lambda_n > 0$ and the eigenfunctions are orthogonal with respect to the weight $r = c_0^{-2}$.

- Superposing the separated solutions that are zero at $t = 0$, we get that

$$u(x, t) = \sum_{n=1}^{\infty} b_n \phi_n(x) \sin(k_n t).$$

- The initial condition for $u_t(x, 0)$ is satisfied if $g(x) = \sum_{n=1}^{\infty} k_n b_n \phi_n(x)$. Using the orthogonality of the eigenfunctions, we get that

$$k_n b_n = \frac{\langle g, \phi_n \rangle_r}{\|\phi_n\|_r^2},$$

or

$$b_n = \frac{\int_0^1 c_0^{-2} g \phi_n dx}{k_n \int_0^1 c_0^{-2} \phi_n^2 dx}. $$
3. The Fourier solution of the initial value problem

\[ u_{tt} = u_{xx} \quad 0 < x < 1, \quad t > 0, \]
\[ u(0, t) = 0, \quad u(1, t) = 0 \quad t > 0, \]
\[ u(x, 0) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1/2 \\ 2(1 - x) & \text{if } 1/2 < x < 1, \end{cases} \]
\[ u_t(x, 0) = 0 \quad 0 \leq x \leq 1, \]

is given by

\[ u(x, t) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n - 1)^2} \sin [(2n - 1)\pi x] \cos [(2n - 1)\pi t] \]

(a) Show that the Fourier series converges to a continuous function. What order of spatial (weak) \(L^2\)-derivatives does \(u(x, t)\) have?

(b) Verify from the Fourier solution that

\[ \int_0^1 [u_t^2(x, t) + u_x^2(x, t)] \, dx = \text{constant} \quad \text{for } -\infty < t < \infty. \]

(c) Use MATLAB (or another program) to compute the partial sum

\[ u_N(x, t) = \frac{8}{\pi^2} \sum_{n=1}^{N} \frac{(-1)^{n+1}}{(2n - 1)^2} \sin [(2n - 1)\pi x] \cos [(2n - 1)\pi t] \]

at \( t = 0.25 \) for \( N = 5 \) and \( N = 50 \).

(d) Use the addition formula for sines to shows that the Fourier solution can be written in the form of the d’Alembert solution as

\[ u(x, t) = F(x - t) + F(x + t) \]

for a suitable function \( F : \mathbb{R} \to \mathbb{R} \). What is \( F \)?

**Solution.**

- (a) We have

\[ \left| \frac{(-1)^{n+1}}{(2n - 1)^2} \sin [(2n - 1)\pi x] \cos [(2n - 1)\pi t] \right| \leq \frac{1}{(2n - 1)^2}, \]
and
\[ \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} < \infty, \]
so the Weierstrass M-test implies that the series of continuous functions converges uniformly to a continuous function.

• We have
\[ u_x(x, t) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \cos[(2n-1)\pi x] \cos[(2n-1)\pi t], \]
so by Parseval’s theorem (and the fact that \( \| \cos m\pi x \|_{L^2}^2 = 1/2 \))
\[ \| u_x(\cdot, t) \|_{L^2}^2 = \frac{32}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos^2[(2n-1)\pi t] \leq \frac{32}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}, \]
meaning that \( u_x(\cdot, t) \in L^2(0, 1) \) for all \( t \in \mathbb{R} \).

• We have the distributional derivative
\[ u_{xx}(x, t) = 8 \sum_{n=1}^{\infty} (-1)^n \sin[(2n-1)\pi x] \cos[(2n-1)\pi t], \]
which does not belong to \( L^2 \) since \( \sum_{n=1}^{\infty} \cos^2[(2n-1)\pi t] \) diverges except for special values of \( t \), such as \( t = 1/2 \). It follows that \( u(\cdot, t) \in H^1(0, 1) \) has one spatial \( L^2 \)-derivative.

• More generally, if we also consider fractional derivatives of order \( s \), then \( u(\cdot, t) \in H^s(0, 1) \) for \( s < 3/2 \).

• (b) We have
\[ u_t(x, t) = -\frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \sin[(2n-1)\pi x] \sin[(2n-1)\pi t], \]
\[ u_x(x, t) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \cos[(2n-1)\pi x] \cos[(2n-1)\pi t]. \]
Parseval’s theorem implies that

\[
\int_0^1 u_t^2(x,t) \, dx = \frac{32}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \sin^2 [(2n-1)\pi t],
\]

\[
\int_0^1 u_x^2(x,t) \, dx = \frac{32}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos^2 [(2n-1)\pi t],
\]

so

\[
\int_0^1 [u_t^2(x,t) + u_x^2(x,t)] \, dx = \frac{32}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}
\]

is a constant independent of time.

- At \( t = 0 \)

\[
\int_0^1 [u_t^2(x,0) + u_x^2(x,0)] \, dx = \int_0^1 u_x^2(x,0) \, dx = 4,
\]

so we get the following sum from the Fourier expansion of the function \( u_x(x,0) \):

\[
\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}.
\]

- (c) The partial sums are shown below. Although the Fourier series converges uniformly, it isn’t rapidly convergent since the solution isn’t a smooth function of \( x \).

- (d) Using the trigonometric identity

\[
\sin A \cos B = \frac{1}{2} [\sin(A - B) + \sin(A + B)],
\]

we get that

\[
u(x,t) = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin [(2n-1)\pi (x-t)]
\]

\[
+ \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin [(2n-1)\pi (x+t)]
\]

\[
= F(x-t) + F(x+t),
\]
where

\[ F(x) = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin[(2n-1)\pi x]. \]

- The function \( 2F : \mathbb{R} \to \mathbb{R} \) is the odd, periodic extension of the initial data, meaning that \( F(-x) = -F(x) \), \( F(x+2) = F(x) \), and

\[ F(x) = \begin{cases} 
  x & \text{if } 0 \leq x \leq 1/2 \\
  1 - x & \text{if } 1/2 < x < 1,
\end{cases} \]
4. Suppose that \( u(x,t) \) is a smooth solution of the wave equation

\[
u_{tt} = c_0^2 \Delta u,
\]

where \( x \in \mathbb{R}^n \), and the wave speed \( c_0 > 0 \) is a constant.

(a) Show that \( u \) satisfies the energy equation

\[
\frac{1}{2} \left( u_t^2 + c_0^2 |\nabla u|^2 \right) - \nabla \cdot \left( c_0^2 u_t \nabla u \right) = 0.
\]

(b) For \( T > 0 \), let \( \Omega_T \subset \mathbb{R}^{n+1} \) be the space-time cone

\[
\Omega_T = \left\{ (x,t) \in \mathbb{R}^{n+1} : |x| < c_0(T-t), 0 < t < T \right\},
\]

and for \( 0 \leq t \leq T \), let \( B(T-t) \) be the spatial cross-section of \( \Omega_T \) at time \( t \)

\[
B(T-t) = \left\{ x \in \mathbb{R}^n : |x| < c_0(T-t) \right\}.
\]

Define

\[
e_T(t) = \frac{1}{2} \int_{B(T-t)} \left( u_t^2 + c_0^2 |\nabla u|^2 \right) dx,
\]

and show that \( e_T(t) \leq e_T(0) \).

(c) Suppose that \( u_1, u_2 \) are smooth solution of the wave equation such that

\[
u_i(x,0) = f_i(x), \quad u_{i t}(x,0) = g_i(x) \quad i = 1, 2
\]

where \( f_1 = f_2, g_1 = g_2 \) in \( |x| \leq c_0 T \), show that \( u_1 = u_2 \) in \( \Omega_T \).

HINT. For (b), apply the divergence theorem in space-time to the equation in (a) over the truncated cone \( \{ (x,t') \in \Omega_T : 0 < t' < t \} \), and note that the space-time normal to the side of the cone \( \Omega_T \) is \( N = (\hat{x}, c_0)/\sqrt{1 + c_0^2} \) where \( \hat{x} = x/|x| \). For (c), consider \( u = u_1 - u_2 \).

Solution.

- (a) Multiplying the wave equation by \( u_t \), we have

\[
u_t u_{tt} - c_0^2 u_t \Delta u = 0.
\]

Using the identities

\[
u_t u_{tt} = \left( \frac{1}{2} u_t^2 \right)_t, \quad u_t \Delta u = \nabla \cdot (u_t \nabla u) - \left( \frac{1}{2} \nabla u \cdot \nabla u \right)_t,
\]

we get that

\[
\frac{1}{2} \left( u_t^2 + c_0^2 |\nabla u|^2 \right)_t - \nabla \cdot \left( c_0^2 u_t \nabla u \right) = 0.
\]
• For $0 < T' < T$, let $\Omega_{T',T}$ denote the truncated cone

$$\Omega_{T',T} = \{(x,t) \in \mathbb{R}^{n+1} : |x| < c_0(T - t) \text{ and } 0 < t < T'\}.$$  

The boundary of $\Omega_{T',T}$

$$\partial \Omega_{T',T} = \Gamma \cup \Sigma \cup \Gamma'$$

consists of the bottom

$$\Gamma = \{(x,0) \in \mathbb{R}^{n+1} : |x| \leq c_0T\}$$

with outward space-time normal $N = (0, -1)$, the side

$$\Sigma = \{(x,t) \in \mathbb{R}^{n+1} : |x| = c_0(T - t) \text{ and } 0 < t < T'\}$$

with outward space-time normal $N = (\hat{x}, c_0)/\sqrt{1 + c_0^2}$, and the top

$$\Gamma' = \{(x,T') \in \mathbb{R}^{n+1} : |x| \leq c_0(T - T')\}$$

with outward space-time normal $N = (0, 1)$.

• Integrating the energy equation over $\Omega_{T',T}$ and applying the divergence theorem, we get that

$$0 = \int_{\Omega_{T',T}} \left\{ \frac{1}{2} \left( u_t^2 + c_0^2 |\nabla u|^2 \right)_t - \nabla \cdot \left( c_0^2 u_t \nabla u \right) \right\} \, dx \, dt$$

$$= \int_{\partial \Omega_{T',T}} \left\{ \frac{1}{2} \left( u_t^2 + c_0^2 |\nabla u|^2 \right) \nu - c_0^2 u_t \nabla u \cdot n \right\} \, dS,$$

where $N = (n, \nu)$ is the outward space-time normal to $\partial \Omega_{T',T}$.

• Splitting the boundary integral into an integral over the bottom, side, and top, we get that

$$e_T(T') = e_T(0) - \frac{c_0}{\sqrt{1 + c_0^2}} \int_{\Sigma} \left\{ \frac{1}{2} \left( u_t^2 + c_0^2 |\nabla u|^2 \right) - c_0 u_t \nabla u \cdot \hat{x} \right\} \, dS,$$

where

$$e_T(T') = \int_{\Gamma'} \frac{1}{2} \left( u_t^2 + c_0^2 |\nabla u|^2 \right) \, dx, \quad e_T(0) = \int_{\Gamma} \frac{1}{2} \left( u_t^2 + c_0^2 |\nabla u|^2 \right) \, dx.$$
Completing the square, and using the fact that $\hat{x}$ is a unit vector, we get that
\[
\frac{1}{2} \left( u_t^2 + c_0^2 |\nabla u|^2 \right) - c_0 u_t \nabla u \cdot \hat{x} = \frac{1}{2} \left( u_t \hat{x} - c_0 \nabla u \right) \cdot \left( u_t \hat{x} - c_0 \nabla u \right) \geq 0.
\]
It follows that $e_T(T') \leq e_T(0)$ for $0 < T' < T$.

A physical interpretation of this inequality is that energy can only propagate out of the cone $\Omega_T$, not into it.

(c) If $u_1$, $u_2$ are two solutions with the same initial data for $u_i$ and $u_{it}$ in $|x| \leq c_0 T$, then $u = u_1 - u_2$ has zero initial data, in $|x| \leq c_0 T$ and therefore $e_T(0) = 0$. The previous inequality implies that $0 \leq e_T(t) \leq e_T(0)$ for $0 < t < T$, so $e_T(t) = 0$. It follows that $u_t = 0$ and $\nabla u = 0$ in $\Omega_T$, meaning that $u = \text{constant}$. Then the initial condition implies that $u = 0$ and $u_1 = u_2$ in $\Omega_T$.

Remark. As this result shows, the solution of the wave equation at some point in space-time can only depend on, or influence, the solution at other points of space-time that can be reached by traveling at speeds less than or equal to $c_0$. 