1. Suppose that $u_1, u_2 : \mathbb{R} \to \mathbb{R}$ are two solutions of the homogeneous Sturm-Liouville equation

$$-(pu')' + qu = 0$$

where $p, q : \mathbb{R} \to \mathbb{R}$ are smooth functions and $p > 0$. If $W = u_1u_2' - u_2u_1'$ is the Wronskian of $u_1, u_2$, show that $pW = \text{constant}$.

2. Compute the Green’s function for the BVP

$$-u'' + u = f(x) \quad 0 < x < 1$$

$$u(0) = 0, \quad u(1) = 0.$$

Write down the integral representation of the solution $u$ in terms of $f$.

3. Compute the Green’s function for the BVP

$$-u'' = f(x) \quad 0 < x < 1$$

$$u(0) + u(1) = 0, \quad u'(0) + u'(1) = 0.$$

Write down the integral representation of the solution $u$ in terms of $f$.

4. Compute the generalized Green’s function $G(x, \xi)$ for the BVP

$$-u'' = \pi^2 u + f(x) \quad 0 < x < 1$$

$$u(0) = 0, \quad u(1) = 0.$$

State the equations that are satisfied by the function

$$u(x) = \int_0^1 G(x, \xi)f(\xi) \, d\xi.$$
5. Consider the Sturm-Liouville equation

\[-(pu')' + qu = \lambda ru, \quad a < x < b\]

where \(p, q, r : [a, b] \to \mathbb{R}\) are smooth functions and \(p(x), r(x) > 0\) for \(a \leq x \leq b\). Show that the change of variables

\[
t = \int_a^x \sqrt{\frac{r(s)}{p(s)}} \, ds, \quad v(t) = [r(x)p(x)]^{1/4} u(x)
\]

transforms this equation into a Sturm-Liouville equation with \(p = r = 1\) of the form

\[-v'' + Qv = \lambda v, \quad 0 < t < c.\]

What are \(c\) and \(Q : [0, c] \to \mathbb{R}\)?