Problem set 6: Solutions

1. Suppose that $u_1, u_2 : \mathbb{R} \to \mathbb{R}$ are two solutions of the homogeneous Sturm-Liouville equation
   $$-(pu')' + qu = 0$$
   where $p, q : \mathbb{R} \to \mathbb{R}$ are smooth functions and $p > 0$. If $W = u_1 u_2' - u_2 u_1'$ is the Wronskian of $u_1, u_2$, show that $pW = \text{constant}$.

Solution.

- Using the ODE, we compute that
  
  $$(pW)' = (u_1 \cdot pu_2' - u_2 \cdot pu_1')'$$
  $$= u_1 \cdot (pu_2)'' + u_1' \cdot pu_2' - u_2 \cdot (pu_1)'' - u_2' \cdot pu_1'$$
  $$= u_1 \cdot qu_2 - u_2 \cdot qu_1$$
  $$= 0$$

  so $pW$ is a constant.
2. Compute the Green’s function for the BVP
\[ -u'' + u = f(x) \quad 0 < x < 1 \]
\[ u(0) = 0, \quad u(1) = 0. \]

Write down the integral representation of the solution \( u \) in terms of \( f \).

**Solution.**

- The Green’s function \( G(x, \xi) \) satisfies
  \[
  -G_{xx} + G = \delta(x - \xi) \quad 0 < x < 1 \\
  G(0, \xi) = 0, \quad G(1, \xi) = 0.
  \]

- Solving the homogeneous ODE for \( 0 < x < \xi \) and \( \xi < x < 1 \) and imposing the appropriate boundary conditions, we get that
  \[
  G(x, \xi) = \begin{cases} 
    A(\xi) \sinh x & \text{if } 0 \leq x < \xi \\
    B(\xi) \sinh(1 - x) & \text{if } \xi < x \leq 1
  \end{cases}
  \]

- Imposition of the continuity of \( G(x, \xi) \) at \( x = \xi \) and the jump-condition
  \[
  -G_x(\xi^+, \xi) + G_x(\xi^-, \xi) = 1
  \]
  gives the equations
  \[
  A(\xi) \sinh \xi - B(\xi) \sinh(1 - \xi) = 0, \\
  A(\xi) \cosh \xi + B(\xi) \cosh(1 - \xi) = 1.
  \]

Using the addition formula
\[
\sinh \xi \cosh(1 - \xi) + \cosh \xi \sinh(1 - \xi) = \sinh 1,
\]
we get the solution
\[
A(\xi) = \frac{\sinh(1 - \xi)}{\sinh 1}, \quad B(\xi) = \frac{\sinh \xi}{\sinh 1},
\]
so the Green’s function is
\[
G(x, \xi) = \frac{\sinh(x_<) \sinh(1 - x_>)}{\sinh 1}
\]
where \( x_< = \min(x, \xi) \), \( x_> = \max(x, \xi) \).

- The Green’s function representation is
  \[
  u(x) = \int_0^1 G(x, \xi) f(\xi) \, d\xi.
  \]
3. Compute the Green’s function for the BVP

\[- u'' = f(x) \quad 0 < x < 1 \]
\[ u(0) + u(1) = 0, \quad u'(0) + u'(1) = 0. \]

Write down the integral representation of the solution \( u \) in terms of \( f \).

Solution.

• The Green’s function \( G(x, \xi) \) satisfies

\[- G_{xx} = \delta(x - \xi), \]
\[ G(0, \xi) + G(1, \xi) = 0, \quad G_x(0, \xi) + G_x(1, \xi) = 0. \]

• The boundary conditions are not separated, so we use general solutions of the homogeneous equation in \( x < \xi \) and \( x > \xi \) to get

\[ G(x, \xi) = \begin{cases} A(\xi) + B(\xi)x & \text{if } 0 \leq x < \xi \\ C(\xi) + D(\xi)(1-x) & \text{if } \xi < x \leq 1 \end{cases} \]

The boundary conditions give

\[ A + C = 0, \quad B - D = 0. \]

The continuity of \( G \) and the jump condition \(-[G_x] = 1 \) give

\[ A + B\xi = C + D(1 - \xi), \quad B + D = 1. \]

• It follows that \( B = D = 1/2 \) and

\[ A = -C = \frac{1}{4} - \frac{1}{2}\xi, \]

so

\[ G(x, \xi) = \begin{cases} 1/4 + (x - \xi)/2 & \text{if } 0 \leq x < \xi \\ 1/4 + (\xi - x)/2 & \text{if } \xi < x \leq 1 \end{cases} = \frac{1}{4} - \frac{1}{2}|x - \xi|. \]

• The Green’s function representation is

\[ u(x) = \int_0^1 G(x, \xi) f(\xi) \, d\xi. \]
4. Compute the generalized Green’s function $G(x, \xi)$ for the BVP
\[-u'' = \pi^2 u + f(x) \quad 0 < x < 1
u(0) = 0, \quad u(1) = 0.\]

State the equations that are satisfied by the function
\[u(x) = \int_0^1 G(x, \xi) f(\xi) \, d\xi.\]

**Solution.**

- A normalized solution of the homogeneous problem, with $L^2$-norm one, is $\phi(x) = \sqrt{2} \sin(\pi x)$.

- The generalized Green’s function $G(x, \xi)$ satisfies
\[-G_{xx} - \pi^2 G = \delta(x - \xi) - 2 \sin(\pi \xi) \sin(\pi x) \quad 0 < x < 1
G(0, \xi) = 0, \quad G(1, \xi) = 0,
\int_0^1 G(x, \xi) \sin(\pi x) \, dx = 0.\]

The right-hand side of the ODE is the projection of the $\delta$-function onto the space orthogonal to $\phi$ to ensure that a solution exists, and the condition $G(\cdot, \xi) \perp \phi$ specifies a unique solution.

- In $x < \xi$ and $x > \xi$, we have
\[-G_{xx} - \pi^2 G = -2 \sin(\pi \xi) \sin(\pi x).\]

- A particular solution of the non-homogeneous ODE
\[-u'' - \pi^2 u = C \sin(\pi x),\]
whose right-hand side is a solution of the homogeneous ODE, is
\[u(x) = \frac{C}{2\pi} x \cos(\pi x),\]
so the general solution is
\[u(x) = A \cos(\pi x) + B \sin(\pi x) + \frac{C}{2\pi} x \cos(\pi x).\]
• Imposing the appropriate boundary conditions on \( G \), we get that
\[
G(x, \xi) = A(\xi) \sin(\pi x) - \frac{1}{\pi} \sin(\pi \xi) x \cos(\pi x)
\]
if \( 0 \leq x < \xi \)
\[
G(x, \xi) = B(\xi) \sin(\pi x) - \frac{1}{\pi} \sin(\pi \xi) x \cos(\pi x)
\]
\[+ \frac{1}{\pi} \sin(\pi \xi) \cos(\pi x) \]
if \( \xi < x \leq 1 \).

• The continuity of \( G \) at \( x = \xi \) implies that
\[
\pi (A - B) = \cos(\pi \xi).
\]
One can verify directly that the jump condition \(-[G_x] = 1\) at \( x = \xi \) gives the same equation, so it is also satisfied, and therefore
\[
G(x, \xi) = B(\xi) \sin(\pi x) - \frac{1}{\pi} \sin(\pi \xi) x \cos(\pi x)
\]
\[+ \frac{1}{\pi} \cos(\pi \xi) \sin(\pi x) \]
if \( 0 \leq x < \xi \)
\[
G(x, \xi) = B(\xi) \sin(\pi x) - \frac{1}{\pi} \sin(\pi \xi) x \cos(\pi x)
\]
\[+ \frac{1}{\pi} \sin(\pi \xi) \cos(\pi x) \]
if \( \xi < x \leq 1 \).

• The orthogonality condition \( G(\cdot, \xi) \perp \phi \) gives, after some algebra,
\[
B = -\frac{1}{\pi} \xi \cos(\pi \xi).
\]

• The generalized Green’s function can then be written as
\[
G(x, \xi) = \frac{1}{\pi} \cos(\pi x_<) \sin(\pi x_>)
\]
\[- \frac{1}{\pi} x_> \cos(\pi x_<) \sin(\pi x_<) - \frac{1}{\pi} x_< \cos(\pi x_<) \sin(\pi x_>)
\]
where \( x_< = \min(x, \xi), \ x_> = \max(x, \xi) \).

• The function \( u(x) \) satisfies
\[
-u'' = \pi^2 u + f(x) - 2 \left( \int_0^1 f(\xi) \sin(\pi \xi) \, d\xi \right) \sin(\pi x),
\]
\[
u(0) = 0, \quad u(1) = 0,
\]
\[
\int_0^1 u(x) \sin(\pi x) \, dx = 0.
\]
5. Consider the Sturm-Liouville equation
\[-(pu')' + qu = \lambda ru, \quad a < x < b\]
where \(p, q, r : [a, b] \to \mathbb{R}\) are smooth functions and \(p(x), r(x) > 0\) for \(a \leq x \leq b\). Show that the change of variables
\[ t = \int_a^x \sqrt{\frac{r(s)}{p(s)}} \, ds, \quad v(t) = [r(x)p(x)]^{1/4} u(x) \]
transforms this equation into a Sturm-Liouville equation with \(p = r = 1\) of the form
\[-v'' + Qv = \lambda v, \quad 0 < t < c.\]
What are \(c\) and \(Q : [0, c] \to \mathbb{R}\)?

Solution.

• This is an exercise in the chain rule. One finds that
\[ Q = q - \frac{(pr)^{1/4}}{r} \left[ p \left( \frac{1}{(pr)^{1/4}} \right)' \right], \quad c = \int_a^b \sqrt{\frac{r(s)}{p(s)}} \, ds. \]

This transformation is called the Liouville transformation, and it shows that every Sturm-Liouville equation can be transformed to a normal form with \(p = r = 1\).