1. (a) Use separation of variables to find the eigenvalues $\lambda$ and eigenfunctions $u(x,y)$ of the Dirichlet Laplacian on the unit square that satisfy

$$- \left( u_{xx} + u_{yy} \right) = \lambda u \quad 0 < x < 1, \ 0 < y < 1$$

$$u(x,0) = 0, \quad u(x,1) = 0 \quad 0 \leq x \leq 1$$

$$u(0,y) = 0, \quad u(1,y) = 0 \quad 0 \leq y \leq 1.$$ 

(b) What is the smallest eigenvalue that is not a simple eigenvalue?

2. (a) Let $\vec{x} = (x,y)$, $\vec{\xi} = (\xi, \eta)$, and $\vec{\xi}^* = (\xi, -\eta)$ where $\eta > 0$. Show that

$$G(\vec{x}, \vec{\xi}) = -\frac{1}{2\pi} \log \left( \frac{|\vec{x} - \vec{\xi}|}{|\vec{x} - \vec{\xi}^*|} \right)$$

is the solution of

$$- \left( G_{xx} + G_{yy} \right) = \delta(\vec{x} - \vec{\xi}) \quad \text{in} \ -\infty < x < \infty, \ y > 0$$

$$G(\vec{x}, \vec{\xi}) = 0 \quad \text{on} \ y = 0.$$ 

(b) Write down the Green’s function representation for the solution $u(x,y)$ of the Dirichlet problem for the Laplacian in the upper half plane

$$u_{xx} + u_{yy} = 0 \quad \text{in} \ -\infty < x < \infty, \ y > 0$$

$$u(x,0) = f(x).$$ 

You can assume that $u(x,y) \to 0$ sufficiently rapidly as $|(x,y)| \to \infty$.

(c) Use the Green’s function representation to show that

$$u(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{(x-t)^2 + y^2} \, dt.$$