1. Consider the BVP

\[ \varepsilon y'' + \sqrt{x} y' + y = 0, \quad 0 < x < 1 \]
\[ y(0) = 2, \quad y(1) = 1 \]

where \(0 < \varepsilon \ll 1\). Where do you expect a boundary layer? Use a dominant balance argument to determine the appropriate inner variable, and find leading order inner, outer, and uniform asymptotic solutions as \(\varepsilon \to 0^+\). You can express your answer in terms of the constant

\[ k = \int_0^\infty \exp \left( -\frac{2}{3} t^{3/2} \right) \, dt = \left( \frac{2}{3} \right)^{1/3} \Gamma \left( \frac{2}{3} \right). \]

Solution

- The coefficients of \(y''\) and \(y'\) have the same sign, corresponding to decay with increasing \(x\), so we expect that a boundary layer will occur at the left endpoint \(x = 0\).

- Introducing inner variables

\[ Y(X; \varepsilon) = y(x; \varepsilon), \quad X = \frac{x}{\delta} \]

in the ODE, we get (remembering to change from \(x\) to \(X\) in the coefficient!)

\[ \frac{\varepsilon}{\delta^2} Y'' + \frac{1}{\sqrt{\delta}} \sqrt{X} Y' + Y = 0, \]

where \(Y'\) denotes the derivative of \(Y\) with respect to \(X\). The first two terms balance if \(\varepsilon/\delta^2 = 1/\sqrt{\delta}\), or

\[ \delta = \varepsilon^{2/3}. \]

We therefore expect a boundary layer at 0 whose thickness is of the order \(\varepsilon^{2/3}\). In that case, the inner equation becomes

\[ Y'' + \sqrt{X} Y' + \varepsilon^{1/3} Y = 0 \]
• **Inner solution.** We expand

\[ Y(X; \epsilon) = Y_0(X) + \epsilon^{1/3}Y_1(X) + O(\epsilon^{2/3}). \]

Then

\[ Y_0'' + \sqrt{X}Y_0' = 0, \quad Y_0(0) = 2. \]

We impose only the BC at the left end-point on the inner solution. Solving this linear first order ODE for \( Y_0' \), we get

\[ Y_0'(X) = C \exp \left( -\frac{2}{3} X^{3/2} \right), \]

where \( C \) is a constant of integration. It follows that

\[ Y_0(X) = 2 + C \int_0^X \exp \left( -\frac{2}{3} t^{3/2} \right) dt. \]

• **Outer solution.** We expand

\[ y(x; \epsilon) = y_0(x) + \epsilon^{1/3}y_1(x) + O(\epsilon^{2/3}). \]

Then, at leading order, we get

\[ \sqrt{x}y_0' + y_0 = 0, \quad y_0(1) = 0. \]

We impose only the BC at the right end-point (without the boundary layer) on the outer solution. Solving this linear first-order IVP for \( y_0 \), we get

\[ y_0(x) = \exp \left[ 2(1 - \sqrt{x}) \right] \]

• **Matching.** We impose the matching condition

\[ \lim_{X \to +\infty} Y_0(X) = \lim_{x \to 0^+} y_0(x) \]

which gives

\[ 2 + kC = \epsilon^2 \]

or

\[ C = \frac{\epsilon^2 - 2}{k}. \]

• The uniform asymptotic solution is

\[ y \sim \exp \left[ 2(1 - \sqrt{x}) \right] + 2 + \left( \frac{\epsilon^2 - 2}{k} \right) \int_0^{x^{2/3}} \exp \left( -\frac{2}{3} t^{3/2} \right) dt - \epsilon^2. \]
2. Consider the BVP

\[ \epsilon y'' + y' + y^3 = 0, \quad 0 < x < 1 \]
\[ y(0) = 0, \quad y(1) = \frac{1}{2} \]

where \( 0 < \epsilon \ll 1 \). Where do you expect a boundary layer? Find leading order inner, outer, and uniform asymptotic solutions as \( \epsilon \to 0^+ \). Would a similar solution work for the BCs \( y(0) = 0, y(1) = 1 \)?

Solution

- For \( \epsilon > 0 \), we expect the boundary layer at \( x = 0 \) since the coefficient of \( y' \) is positive.

- **Inner solution.** We let
  \[ y(x; \epsilon) = Y(X; \epsilon), \quad X = \frac{x}{\epsilon} \]
  and expand
  \[ Y(X; \epsilon) = Y_0(X) + \epsilon Y_1(X) + O(\epsilon^2). \]
  Then
  \[ Y''_0 + Y'_0 = 0, \quad Y_0(0) = 0, \]
  which gives
  \[ Y_0(X) = C \left( 1 - e^{-X} \right). \]

- **Outer solution.** We expand
  \[ y(x; \epsilon) = y_0(x) + \epsilon y_1(x) + O(\epsilon^2). \]
  Then, at leading order, we get
  \[ y'_0 + y_0^3 = 0, \quad y_0(1) = \frac{1}{2}. \]
  The solution of the ODE is
  \[ y_0(x) = \frac{1}{\sqrt{2x + A}} \]
  and the BC implies that
  \[ A = 2. \]
• **Matching.** We impose the matching condition

\[
\lim_{X \to +\infty} Y_0(X) = \lim_{x \to 0^+} y_0(x)
\]

which gives

\[
C = \frac{1}{\sqrt{2}}
\]

• The uniform asymptotic solution is

\[
y \sim \frac{1}{\sqrt{2x + 2}} - \frac{1}{\sqrt{2}} \exp\left(-\frac{x}{\epsilon}\right).
\]

• If, instead, \(y(1) = 1\), then the outer solution is

\[
y_0(x) = \frac{1}{\sqrt{2x - 1}}.
\]

This solution blows up at \(x = 1/2\), so the asymptotic solution breaks down. Further analysis would be required to see if the BVP has a solution at all in this case.
3. Consider the BVP
\[ \epsilon y'' - \frac{y'}{1 + 2x} - \frac{1}{y} = 0, \quad 0 < x < 1 \]
\[ y(0) = 3, \quad y(1) = 3 \]
where \( 0 < \epsilon \ll 1 \). Where do you expect a boundary layer? Find leading order inner, outer, and uniform asymptotic solutions as \( \epsilon \to 0^+ \).

Solution

- We expect the boundary layer at the right end-point \( x = 1 \) since the coefficient of \( y' \) is negative.

- **Inner solution.** We let 
  \[ y(x; \epsilon) = Y(X; \epsilon), \quad X = \frac{1 - x}{\epsilon} \]
  and expand
  \[ Y(X; \epsilon) = Y_0(X) + \epsilon Y_1(X) + O(\epsilon^2). \]
  Note that \( X = 0 \) corresponds to \( x = 1 \),
  \[ \frac{d}{dx} = -\frac{1}{\epsilon} \frac{d}{dX}, \]
  and \( x = 1 + O(\epsilon) \) in the coefficient. Therefore, at leading order, we get
  \[ Y_0'' + \frac{1}{3} Y'_0 = 0, \quad Y_0(0) = 3. \]
  The solution is
  \[ Y_0(X) = 3 + C \left(1 - e^{-X/3}\right) \]
  where \( C \) is a constant of integration.

- **Outer solution.** We expand
  \[ y(x; \epsilon) = y_0(x) + \epsilon y_1(x) + O(\epsilon^2). \]
  Then, at leading order, we get
  \[ \frac{y_0'}{1 + 2x} + \frac{1}{y_0} = 0 = 0, \quad y_0(0) = 3. \]
The solution of this first-order nonlinear ODE (by separation of variables) is

$$y_0(x) = \sqrt{A - 2(x + x^2)},$$

and the BC implies that

$$A = 9.$$

The outer solution is well-defined and nonzero in $0 \leq x \leq 1$.

- **Matching.** We impose the matching condition

$$\lim_{X \to +\infty} Y_0(X) = \lim_{x \to 1^-} y_0(x)$$

which gives

$$C = \sqrt{5} - 3$$

- The uniform asymptotic solution is

$$y \sim \sqrt{9 - 2(x + x^2)} + \left(3 - \sqrt{5}\right) \left\{1 - \exp\left[-\left(1 - \frac{x}{3\epsilon}\right)\right]\right\}.$$