1. Consider the following scalar initial value problem for $x(t; \epsilon)$:

$$\dot{x} = \epsilon x \sin^2 t, \quad x(0; \epsilon) = 1.$$ 

(a) Write down the averaged equation $\dot{y} = \epsilon \bar{f}(y)$ and solve for $y(t; \epsilon)$.

(b) Solve the full problem and verify explicitly that the averaged solution $y(t; \epsilon)$ approximates the exact solution $x(t; \epsilon)$ up to an error that is of the order $\epsilon$ over times of the order $1/\epsilon$.

2. Consider the weakly damped van der Pol equation with small external forcing at the linearized frequency:

$$\ddot{y} + \epsilon \left( y^2 - 1 \right) \dot{y} + y = \epsilon k \cos t.$$ 

(a) Put the equation in periodic standard form $\dot{x} = \epsilon f(x, t)$ where $x = (x_1, x_2)$ and

$$y = x_1 \cos t + x_2 \sin t, \quad \dot{y} = -x_1 \sin t + x_2 \cos t.$$ 

(b) Find the averaged equation $\dot{y} = \epsilon \bar{f}(y)$.

(c) Does the forced van der Pol equation have $2\pi$-periodic solutions? If so, what can you say about their stability?
3. Consider the Sturm-Liouville ODE

\[ y'' + \lambda q(x)y = 0 \]

where \( q(x) > 0 \) is a smooth, positive coefficient function and \( \lambda \) is a large positive parameter

(a) Show that the WKB approximation of the solution is given by

\[ y(x; \lambda) \sim A q(x)^{-1/4} \cos \left( \sqrt{\lambda} S(x) \right) + B q(x)^{-1/4} \sin \left( \sqrt{\lambda} S(x) \right) \]

as \( \lambda \to \infty \), where \( A, B \) are arbitrary constants and

\[ S(x) = \int_0^x \sqrt{q(\xi)} \, d\xi. \]

(b) Obtain the WKB approximation, valid as \( n \to \infty \), of the eigenvalues \( \lambda = \lambda_n \), where \( n = 1, 2, 3, \ldots \), for the following eigenvalue problem

\[ y'' + \lambda (x + \pi)^4 y = 0 \quad 0 < x < \pi, \]

\[ y(0) = 0, \quad y(\pi) = 0. \]

Compare your results with the following numerical values:

\[ \lambda_1 = 0.00174401, \quad \lambda_2 = 0.00734865, \quad \lambda_3 = 0.0167524, \]
\[ \lambda_5 = 0.0469006, \quad \lambda_{10} = 0.188305, \quad \lambda_{40} = 3.01668. \]