1. Suppose that $\Omega$ is a connected, bounded open set with $C^1$-boundary that satisfies the interior sphere condition at every point of its boundary. Use a maximum principle argument to prove that a solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of the Neumann problem

$$\begin{align*}
-\Delta u &= f & \text{in $\Omega$}, \\
\frac{\partial u}{\partial \nu} &= g & \text{on $\partial \Omega$},
\end{align*}$$

is unique up to an arbitrary additive constant.

2. Suppose that $\Omega$ is a bounded open set and $0 < \alpha \leq 1$. If $u : \Omega \to \mathbb{R}$, let

$$\begin{align*}
|u|_0 &= \sup_\Omega |u|, &
|u|_1 &= \sup_\Omega |Du|, \\
[u]_{1,\alpha} &= \sup_{x,y \in \Omega \atop x \neq y} \frac{|Du(x) - Du(y)|}{|x-y|^\alpha}.
\end{align*}$$

Prove the following interpolation inequality: For any $\epsilon > 0$ there exists a constant $C_\epsilon$ such that

$$|u|_1 \leq C_\epsilon |u|_0 + \epsilon [u]_{1,\alpha} \quad \text{for all } u \in C^{1,\alpha}(\overline{\Omega}).$$

HINT. Assume the inequality is false and use a compactness argument to derive a contradiction.

3. Suppose that $f \in C_c^\infty(\mathbb{R}^n)$ and let $u = \Gamma \ast f$ be the Newtonian potential of $f$. If $\text{spt} \, f \subset B_R(0)$, prove that

$$\sup_{B_R(0)} (|u| + |Du|) \leq CR^2 \sup |f|$$

where $C$ is a constant depending only on $n$.

4. Give an example of a function $f \in C_c(\mathbb{R}^2)$ such that there is no solution $u \in C^2(\mathbb{R}^2)$ of Poisson’s equation

$$u_{xx} + u_{yy} = f(x,y).$$