1. Look for self-similar, spherically symmetric solutions of the heat equation
\[ u_t = \Delta u \quad x \in \mathbb{R}^n, \ t > 0 \]
of the form
\[ u(x, t) = \frac{1}{t^\alpha} f \left( \frac{|x|}{t^\beta} \right). \]
(a) Determine the values of $\alpha, \beta \in \mathbb{R}$ for which there is a solution of this form and find the ODE satisfied by $f(\xi)$.
(b) Find the values of $\alpha, \beta$ if
\[ \lim_{t \to 0^+} \int_{\mathbb{R}^n} u(x, t) \, dx = 1. \]
Solve the ODE explicitly in this case, and show that you get the fundamental solution of the heat equation.

2. (a) Solve the IVP for the Schrödinger equation
\[ iu_t = -\Delta u \quad x \in \mathbb{R}^n, \ t \in \mathbb{R}, \]
\[ u(x, 0) = f(x) \]
where $f \in \mathcal{S}$, and show that $u \in C^\infty (\mathbb{R}; \mathcal{S})$ is given by
\[ u(x, t) = \int \Gamma(x - y, t) f(y) \, dy \]
where
\[ \Gamma(x, t) = \frac{1}{(4\pi it)^{n/2}} e^{-|x|^2/4it}. \]
(b) Prove that for any $s \in \mathbb{R}$, the solution operators of the Schrödinger equation $\{e^{i t \Delta} : t \in \mathbb{R}\}$ form a strongly continuous unitary group on $H^s$.

3. Suppose that $s > n/2$. Prove that $H^s(\mathbb{R}^n)$ is an algebra, meaning that $fg \in H^s$ if $f, g \in H^s$, and there is a constant $C_s$ such that
\[ \|fg\|_{H^s} \leq C_s \|f\|_{H^s} \|g\|_{H^s} \quad \text{for all } f, g \in H^s. \]
HINT. Use the convolution theorem and the inequality
\[ (1 + |k + \xi|^2)^{s/2} \leq 2^s \left[ (1 + |k|^2)^{s/2} + (1 + |\xi|^2)^{s/2} \right]. \]