Problem set 1

Math 218B, Winter 2010

1. Look for self-similar, spherically symmetric solutions of the heat equation

$$u_t = \Delta u \qquad x \in \mathbb{R}^n,\ t > 0$$

of the form

$$u(x,t) = \frac{1}{t^\alpha} f\left(\frac{|x|}{t^\beta}\right).$$

(a) Determine the values of $\alpha, \beta \in \mathbb{R}$ for which there is a solution of this form and find the ODE satisfied by $f(\xi)$.

(b) Find the values of α, β if

$$\lim_{t\to 0^+} \int_{\mathbb{R}^n} u(x,t)\,dx = 1.$$

Solve the ODE explicitly in this case, and show that you get the fundamental solution of the heat equation.

2. (a) Solve the IVP for the Schrödinger equation

$$\begin{aligned} &iu_t = -\Delta u \qquad x \in \mathbb{R}^n,\ t \in \mathbb{R}, \\ &u(x,0) = f(x) \end{aligned}$$

where $f \in \mathcal{S}$, and show that $u \in C^\infty(\mathbb{R}; \mathcal{S})$ is given by

$$u(x,t) = \int \Gamma(x-y,t) f(y)\,dy$$

where

$$\Gamma(x,t) = \frac{1}{(4\pi i t)^{n/2}} e^{-|x|^2/4it}.$$

(b) Prove that for any $s \in \mathbb{R}$, the solution operators of the Schrödinger equation $\{e^{it\Delta} : t \in \mathbb{R}\}$ form a strongly continuous unitary group on H^s.

3. Suppose that $s > n/2$. Prove that $H^s(\mathbb{R}^n)$ is an algebra, meaning that $fg \in H^s$ if $f, g \in H^s$, and there is a constant C_s such that

$$\|fg\|_{H^s} \le C_s \|f\|_{H^s} \|g\|_{H^s} \qquad \text{for all } f, g \in H^s.$$

Hint. Use the convolution theorem and the inequality

$$\left(1 + |k+\xi|^2\right)^{s/2} \le 2^s \left[\left(1+|k|^2\right)^{s/2} + \left(1+|\xi|^2\right)^{s/2}\right].$$