Problem set 2
Math 218B, Winter 2010

You can use standard theorems about semigroups or the Fourier transform

1. The linearized Kuramoto-Sivashinsky equation for $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ describes systems with long-wave instability and short-wave stability:

$$u_t + \Delta u + \Delta^2 u = 0.$$

Show that this equation generates a strongly continuous semigroup $\{\mathrm{T}(t) : t \geq 0\}$ on $L^2(\mathbb{R}^n)$. Find constants $M \geq 1$, $a \in \mathbb{R}$ such that

$$\|\mathrm{T}(t)\|_{\mathcal{B}(L^2)} \leq Me^{at} \qquad \text{for all } t \geq 0.$$

2. Write the wave equation

$$u_{tt} = \Delta u$$

as a first-order system

$$u_t = v, \qquad v_t = \Delta u$$

for $(u, v) = (u, u_t)$. Show that this equation generates a unitary group on $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

3. Write the semilinear heat equation

$$u_t = \Delta u - \lambda u^3, \qquad u(0) = f$$

as an integral equation

$$u(t) = \mathrm{T}(t)f - \lambda \int_0^t \mathrm{T}(t-s)u^3(s)\, ds$$

where $\mathrm{T}(t) = e^{t\Delta}$. Prove that for suitable values of p and $T > 0$ there is a unique solution

$$u \in C\left([0, T]; L^p(\mathbb{R}^n)\right).$$