Problem set 2
Math 218B, Winter 2010

You can use standard theorems about semigroups or the Fourier transform

1. The linearized Kuramoto-Sivashinsky equation for \( u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R} \) describes systems with long-wave instability and short-wave stability:

\[
  u_t + \Delta u + \Delta^2 u = 0.
\]

Show that this equation generates a strongly continuous semigroup \( \{T(t) : t \geq 0\} \) on \( L^2(\mathbb{R}^n) \). Find constants \( M \geq 1, a \in \mathbb{R} \) such that

\[
  \|T(t)\|_{B(L^2)} \leq M e^{at} \quad \text{for all } t \geq 0.
\]

2. Write the wave equation

\[
  u_{tt} = \Delta u
\]

as a first-order system

\[
  u_t = v, \quad v_t = \Delta u
\]

for \( (u, v) = (u, u_t) \). Show that this equation generates a unitary group on \( H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \).

3. Write the semilinear heat equation

\[
  u_t = \Delta u - \lambda u^3, \quad u(0) = f
\]

as an integral equation

\[
  u(t) = T(t)f - \lambda \int_0^t T(t-s)u^3(s) \, ds
\]

where \( T(t) = e^{t\Delta} \). Prove that for suitable values of \( p \) and \( T > 0 \) there is a unique solution

\[
  u \in C([0,T]; L^p(\mathbb{R}^n)).
\]