PROBLEM SET 3
Math 218B, Winter 2010

1. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and denote by $\{w_k : k \in \mathbb{N}\}$ an L^2-orthonormal set of eigenfunctions of the Dirichlet Laplacian on Ω, where

$$-\Delta w_k = \lambda_k w_k, \qquad w_k \in H_0^1(\Omega)$$

and $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$. Suppose that $\lambda \in \mathbb{R}$, $g \in L^2(\Omega)$ and $f \in H^{-1}(\Omega)$.
(a) Find an explicit expression for the weak solution

$$u \in L^2_{\mathrm{loc}}\left(0, \infty; H_0^1(\Omega)\right) \cap C\left(0, \infty; L^2(\Omega)\right)$$

of the IBVP

$$\begin{aligned} &u_t = \Delta u + \lambda u + f(x), \\ &u(x,0) = g(x), \qquad u = 0 \quad \text{on } \partial\Omega \end{aligned}$$

in terms of the eigenfunctions w_k.
(b) If $\lambda < \lambda_1$, show that the solution $u(t)$ converges as $t \to \infty$ to the solution $\bar{u}$ of the time-independent equation, and estimate the rate at which $\|u(t) - \bar{u}\|$ approaches zero in a suitable norm.

2. Let Ω be a bounded open set in $\mathbb{R}^n$. Give a weak formulation for the following IBVP for a fourth-order parabolic PDE for $u(x,t)$:

$$\begin{aligned} &u_t = -\Delta^2 u + f(x,t) && x \in \Omega \text{ and } 0 < t < T, \\ &u = 0, \quad Du \cdot \nu = 0 && \text{on } \partial\Omega \text{ and } 0 < t < T, \\ &u(x,0) = g(x) && x \in \Omega \text{ and } t = 0, \end{aligned}$$

and show that there is a unique weak solution.

3. (a) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $T > 0$. Consider the semilinear IBVP

$$u_t = \Delta u - u^{2p-1}, \qquad u(x,0) = g(x), \qquad u = 0 \quad \text{on } \partial\Omega,$$

where $g \in L^2(\Omega)$ and $p \in \mathbb{N}$. Use a Galerkin method to prove that there exists a unique weak solution

$$u \in L^2\left(0, T; H_0^1(\Omega)\right) \cap L^{2p}\left(0, T; L^{2p}(\Omega)\right)$$

with $u \in C\left([0, T]; L^2(\Omega)\right)$.

(b) Show that, in general, global in time smooth solutions do not exist for the IBVP on $0 < x < 1$

$$u_t = u_{xx} + u^3, \qquad u(x, 0) = g(x), \qquad u = 0 \quad \text{at } x = 0, 1.$$

HINT. See §6.6 of the notes for help on (a). For(b), consider the first Fourier sine coefficient $c(t) = \int_0^1 \sin(\pi x) u(x, t)\, dx$.