Problem set 3  
Math 218B, Winter 2010

1. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and denote by $\{w_k : k \in \mathbb{N}\}$ an $L^2$-orthonormal set of eigenfunctions of the Dirichlet Laplacian on $\Omega$, where

$$-\Delta w_k = \lambda_k w_k, \quad w_k \in H^1_0(\Omega)$$

and $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$. Suppose that $\lambda \in \mathbb{R}$, $g \in L^2(\Omega)$ and $f \in H^{-1}(\Omega)$.

(a) Find an explicit expression for the weak solution $u \in L^2_{\text{loc}}(0, \infty; H^1_0(\Omega)) \cap C(0, \infty; L^2(\Omega))$ of the IBVP

$$u_t = \Delta u + \lambda u + f(x),$$

$$u(x, 0) = g(x), \quad u = 0 \text{ on } \partial \Omega$$

in terms of the eigenfunctions $w_k$.

(b) If $\lambda < \lambda_1$, show that the solution $u(t)$ converges as $t \to \infty$ to the solution $\bar{u}$ of the time-independent equation, and estimate the rate at which $\|u(t) - \bar{u}\|$ approaches zero in a suitable norm.

2. Let $\Omega$ be a bounded open set in $\mathbb{R}^n$. Give a weak formulation for the following IBVP for a fourth-order parabolic PDE for $u(x, t)$:

$$u_t = -\Delta^2 u + f(x, t) \quad x \in \Omega \text{ and } 0 < t < T,$$

$$u = 0, \quad Du \cdot \nu = 0 \quad \text{on } \partial \Omega \text{ and } 0 < t < T,$$

$$u(x, 0) = g(x) \quad x \in \Omega \text{ and } t = 0,$$

and show that there is a unique weak solution.

3. (a) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $T > 0$. Consider the semilinear IBVP

$$u_t = \Delta u - u^{2p-1}, \quad u(x, 0) = g(x), \quad u = 0 \text{ on } \partial \Omega,$$

where $g \in L^2(\Omega)$ and $p \in \mathbb{N}$. Use a Galerkin method to prove that there exists a unique weak solution

$$u \in L^2 \left(0, T; H^1_0(\Omega)\right) \cap L^{2p} \left(0, T; L^{2p}(\Omega)\right)$$
with \( u \in C ([0, T]; L^2(\Omega)) \).

(b) Show that, in general, global in time smooth solutions do not exist for the IBVP on \( 0 < x < 1 \)

\[
 u_t = u_{xx} + u^3, \quad u(x, 0) = g(x), \quad u = 0 \quad \text{at} \ x = 0, 1.
\]

**HINT.** See §6.6 of the notes for help on (a). For (b), consider the first Fourier sine coefficient \( c(t) = \int_0^1 \sin(\pi x) u(x, t) \, dx \).