Additional Problem
Math 218B, Winter, 2020
Due: Tue, Feb 25

Consider the following initial value problem for the inviscid Burgers equation with initial data $u_0 \in C_c^\infty(0,1)$:

$$\begin{aligned} &u_t + uu_x = 0 \qquad 0 < x < 1, \quad t > 0, \\ &u(x,0) = u_0(x), \qquad 0 < x < 1. \end{aligned} \tag{1}$$

(a) Suppose that $u(x,t)$ is a smooth solution of (1) and let

$$c(t) = \int_0^1 xu(x,t)\,dx.$$

Show that

$$c_t \geq \frac{3}{2}c^2,$$

and deduce that if $c_0 = \int_0^1 xu_0(x)\,dx > 0$, then smooth solutions of (1) cannot be continuoued past the time $t = \tilde{T} > 0$ where

$$\tilde{T} = \frac{2}{3c_0}.$$

(b) Solve (1) by the method of characteristics and show that the solution remains bounded but $\|u_x(\cdot,t)\|_{L^\infty} \to \infty$ as $t \to T_*^+$ where

$$T_* = \frac{1}{\max\{-u_0'(x)\}} < \tilde{T}.$$

(So, in fact, neither $u(\cdot,t)$ or $c(t)$ in (a) blows up.)