1. Compute the derivatives of the following functions. You do not need to simplify your answers.

(a) \( \frac{2x + 5}{\sqrt{x^4 + 3}} \)

(b) \( e^{3x+1} \ln(x^2 + 1) \)

(c) \( \tan(cos x + \sin x) \)

(d) \( \sin^{-1} \left( \sqrt{2x} \right) \)

Solution.

• (a)

\[
\frac{d}{dx} \frac{2x + 5}{\sqrt{x^4 + 3}} = \frac{2 \cdot \frac{\sqrt{x^4 + 3} - \frac{1}{2\sqrt{x^4 + 3}} \cdot 4x^3 \cdot (2x + 5)}{x^4 + 3}}
\]

• (b)

\[
\frac{d}{dx} e^{3x+1} \ln(x^2 + 1) = e^{3x+1} \cdot 3 \cdot \ln(x^2 + 1) + e^{3x+1} \cdot \frac{1}{x^2 + 1} \cdot 2x
\]

• (c)

\[
\frac{d}{dx} \tan(cos x + \sin x) = \sec^2(cos x + \sin x) \cdot (- \sin x + \cos x)
\]

• (d)

\[
\frac{d}{dx} \sin^{-1} \left( \sqrt{2x} \right) = \frac{1}{\sqrt{1 - 2x}} \cdot \frac{1}{2\sqrt{2x}} \cdot 2
\]
2. Evaluate the following limits or say if they do not exist (using any method you want):

(a) \[ \lim_{x \to 0} \frac{1 - \cos(2x)}{xe^x - x} \]

(b) \[ \lim_{x \to 0} \left\{ \sin x \left[ \frac{1}{x} - \frac{1}{\sin(2x)} \right] \right\} \]

(c) \[ \lim_{x \to \infty} \left( \ln x e^{-x} \right) \]

(d) \[ \lim_{x \to 1} \frac{\sin x}{x} \]

Solution.

• (a) This is an indeterminate limit \((0/0)\), and applying l'Hôpital's rule twice, we get

\[ \lim_{x \to 0} \frac{1 - \cos(2x)}{xe^x - x} = \lim_{x \to 0} \frac{2 \sin(2x)}{xe^x + e^x - 1} = \lim_{x \to 0} \frac{4 \cos(2x)}{xe^x + 2e^x} = 2. \]

• (b) Using \(\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1\), we get

\[ \lim_{x \to 0} \left\{ \sin x \left[ \frac{1}{x} - \frac{1}{\sin(2x)} \right] \right\} = \lim_{x \to 0} \left\{ \frac{\sin x}{x} - \frac{\sin x}{x} \cdot \frac{2x}{\sin(2x)} \cdot \frac{1}{2} \right\} = 1 - 1 \cdot 1 \cdot \frac{1}{2} = \frac{1}{2}. \]

• (c) Since \(\ln x \to \infty\) and \(e^x \to \infty\) as \(x \to \infty\), we can apply l'Hôpital's rule to get

\[ \lim_{x \to \infty} \left( \ln x e^{-x} \right) = \lim_{x \to \infty} \frac{\ln x}{e^x} = \lim_{x \to \infty} \frac{1}{e^x} = 0. \]

• (d) By the continuity of \(\sin x\),

\[ \lim_{x \to 1} \frac{\sin x}{x} = \sin 1. \]
3. A 15 ft ladder is leaning against a wall. If the base of the ladder is pushed toward the wall at a speed of 2 ft/sec, at what speed is the top of the ladder moving up the wall when the base of the ladder is 6 ft from the wall?

Solution.

- Let $y(t)$ be the height of the top of the ladder and $x(t)$ the distance of the base of the ladder from the wall at time $t$. Then, by the Pythagorean theorem,
  \[ x^2 + y^2 = 15^2. \]
  If $x = 6$, then $y = \sqrt{15^2 - 6^2} = 3\sqrt{21}$.

- Differentiation with respect to $t$ gives
  \[ 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0, \]
  so
  \[ \frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}. \]

- When $x = 6, y = 3\sqrt{21}$, and $dx/dt = -2$, we get
  \[ \frac{dy}{dt} = \frac{4}{\sqrt{21}} \text{ ft/sec}. \]
4. A pile of sand in the shape of a cone whose radius is twice its height is growing at a rate of 5 cubic meters per second. How fast is its height increasing when the radius is 20 meters? Hint. The volume of a cone of radius \( r \) and height \( h \) is \( V = \frac{1}{3} \pi r^2 h \).

Solution.

- If \( r = 2h \), then
  \[ V = \frac{4}{3} \pi h^3. \]

- Differentiation with respect to time \( t \) gives
  \[ \frac{dV}{dt} = 4\pi h^2 \frac{dh}{dt}. \]

- When \( r = 20 \), \( h = 10 \), and \( dV/dt = 5 \), we get
  \[ \frac{dh}{dt} = \frac{1}{80\pi} \text{ meters/second}. \]
5. State the natural domain of the function

\[ y = \frac{\ln x}{x^2}. \]

Sketch the graph, identify where the graph is increasing/decreasing, the local extrema, where the graph is concave up/concave down, and the inflection points.

Solution.

- The domain is \((0, \infty)\), where \(x > 0\).
- We have
  \[ y' = \frac{1 - 2 \ln x}{x^3}. \]
- Increasing \((y' > 0)\) for \(\ln x < 1/2\) or \(0 < x < \sqrt{e}\).
- Decreasing \((y' < 0)\) for \(\ln x > 1/2\) or \(x > \sqrt{e}\).
- Global maximum at \((x, y) = (\sqrt{e}, 1/2e)\).
- We have
  \[ y'' = \frac{6 \ln x - 5}{x^6}. \]
- Concave up \((y'' > 0)\) for \(\ln x > 5/6\) or \(x > e^{5/6}\).
- Concave down \((y'' < 0)\) for \(\ln x < 5/6\) or \(0 < x < e^{5/6}\).
- Inflection point at \((x, y) = (e^{5/6}, (5/6)e^{-5/3})\).
- The graph is on the next page \((\sqrt{e} \approx 1.65, e^{5/6} \approx 2.30)\).
6. A cricket ball is projected directly upward from the ground with an initial velocity of 112 ft/s. Assuming that the acceleration due to gravity is 32 ft/sec², derive an equation for the height \( s(t) \) of the ball above the ground after \( t \) seconds. When does the ball hit the ground?

**Solution.**

- Measuring height upwards, the acceleration of the ball is given by
  \[
  \frac{d^2s}{dt^2} = -32.
  \]
- It follows that
  \[
  \frac{ds}{dt} = -32t + v_0
  \]
  where \( v_0 \) is a constant.
- At \( t = 0 \), the velocity \( ds/dt \) is 112, so \( v_0 = 112 \).
- It also follows that
  \[
  s = -16t^2 + v_0t + s_0
  \]
  where \( s_0 \) is a constant.
- At \( t = 0 \) the height of the ball is 0, so \( s_0 = 0 \) and
  \[
  s(t) = -16t^2 + 112t.
  \]
- Since \( s = 16t(7 - t) \), the ball hits the ground (\( s = 0 \)) after 7 seconds.
7. A one meter high fence is eight meters in front of a high wall. Find the minimum length of a ladder resting on the fence whose foot is in front of the fence and whose top reaches the wall.

Solution.

- Let \( y \) be the height of the top of the ladder at the wall, \( x \) the distance of the foot of the ladder from the wall, and \( a \) the distance of the foot of the ladder from the fence. Then \( x = 8 + a \) and, by similar triangles, \( y/x = 1/a \), so
  \[
  y = \frac{x}{x-8}.
  \]

- Minimizing the length \( L \) of the ladder is the same as minimizing the square \( S = L^2 \) of the length, which is
  \[
  S = x^2 + y^2.
  \]

- It follows that we want to find the global minimum of
  \[
  S(x) = x^2 + \left( \frac{x}{x-8} \right)^2 \quad \text{on } 8 \leq x < \infty.
  \]

- We compute that
  \[
  S'(x) = 2x + 2 \left( \frac{x}{x-8} \right) \left[ \frac{(x-8) - x}{(x-8)^2} \right]
  = 2x - \frac{16x}{(x-8)^3}
  = \frac{2x[(x-8)^3 - 8]}{(x-8)^3}
  \]

- There is one critical point with \( S'(x) = 0 \) and \( x > 8 \) when \( (x-8)^3 = 8 \) or \( x = 10 \). Moreover, \( S'(x) < 0 \) if \( 8 \leq x < 10 \), so \( S(x) \) is decreasing, and \( S'(x) > 0 \) if \( x > 10 \), so \( S(x) \) is increasing. It follows that \( S(x) \) has a global minimum at \( x = 10 \).

- At \( x = 10 \), we have \( y = 5 \) and \( S = 125 \). The minimum length \( \sqrt{S} \) of the ladder is therefore \( 5\sqrt{5} \) meters.
8. Define a function $f(x)$ with domain $(-\infty, \infty)$ by

$$f(x) = \begin{cases} 
x^2 & \text{if } x \leq 1 \\
Ax + B & \text{if } 1 < x < 2 \\
-2x^2 & \text{if } x \geq 2.
\end{cases}$$

(a) Determine the constants $A$ and $B$ so that $f(x)$ is continuous everywhere.

(b) Is this function differentiable everywhere?

(c) Sketch the graph $y = f(x)$ in that case and determine the range of $f$.

Solution.

- (a) To get continuity, we need $Ax + B = x^2$ at $x = 1$ and $Ax + B = -2x^2$ at $x = 2$, or

$$A + B = 1, \quad 2A + B = -8.$$ 

The solution of these equations is

$$A = -9, \quad B = 10.$$ 

- (b) The function is not differentiable at $x = 1$ or $x = 2$ since the left and right derivatives are different: $f'(1^+) = -9$, $f'(1^-) = 2$, $f'(2^+) = -8$, $f'(2^-) = -9$.

- (c) The range of $f$ is $(-\infty, \infty)$. The graph is omitted.
9. Find the equation of the tangent line to the curve

\[(x - y)^3 = x^2 - y^2 - 2\]

at the point (2, 1). At which point does this tangent line cross the x-axis and at what angle?

**Solution.**

- Implicit differentiation gives
  
  \[3(x - y)^2 \left(1 - \frac{dy}{dx}\right) = 2x - 2y \frac{dy}{dx}.\]

- At the point (2, 1) we get that
  
  \[\frac{dy}{dx} = -1.\]

- The equation of the tangent line is \[y - 2 = -(x - 1)\] or
  
  \[y = 3 - x.\]

- The tangent line crosses the x-axis at \((x, y) = (3, 0)\). It has slope -1, so the angle is \(3\pi/4\).