1. [20%] Say if the following statements are true or false. (For this question only, you don’t have to explain your answers).
   (a) If \( \lim_{x \to c^+} f(x) = 3 \) and \( \lim_{x \to c^-} f(x) = 3.001 \), then \( \lim_{x \to c} f(x) \) is close to 3.
   (b) If \( \lim_{x \to c} f(x) \) exists, then \( f(x) \) is continuous at \( c \).
   (c) If \( f(x) \) is continuous at \( c \), then \( \lim_{x \to c} f(x) \) exists.
   (d) If \( \lim_{x \to 0^+} f(x) = 0 \), then \( f(x) > 0 \) for all \( x > 0 \) that are sufficiently close to 0.

**Solution.**

- (a) False. (If the left and right limits are different, then the limit does not exist.)
- (b) False. (The function \( f(x) \) need not be defined at \( c \), or its value may be different from the limit.)
- (c) True. (The existence of the limit is part of the definition of continuity.)
- (d) False. (For example, \( \lim_{x \to 0^+} (-x) = 0 \) but \( -x < 0 \) for all \( x > 0 \).)
2. [30%] Evaluate the following limits or say if they do not exist.

(a) \( \lim_{x \to 2} \frac{2x^2 + 1}{11 - x^3} \)

(b) \( \lim_{x \to 0} \frac{\sqrt{5x + 4} - 2}{x} \)

(c) \( \lim_{x \to 0} \frac{\sin(1/x)}{x} \)

(d) \( \lim_{x \to 3^-} \frac{x^2 - 2x - 3}{|x - 3|} \)

Solution.

• (a) Since the limit of the denominator is non-zero, we have

\[
\lim_{x \to 2} \frac{2x^2 + 1}{11 - x^3} = \frac{2 \cdot 2^2 + 1}{11 - 2^3} = 3.
\]

• (b) Using the difference of two squares to simplify the square root, we get

\[
\lim_{x \to 0} \frac{\sqrt{5x + 4} - 2}{x} = \lim_{x \to 0} \frac{\left(\sqrt{5x + 4} - 2\right) \left(\sqrt{5x + 4} + 2\right)}{x \left(\sqrt{5x + 4} + 2\right)}
\]

\[
= \lim_{x \to 0} \frac{5x}{x \left(\sqrt{5x + 4} + 2\right)}
\]

\[
= \lim_{x \to 0} \frac{5}{\sqrt{5x + 4} + 2}
\]

\[
= \frac{5}{4}.
\]

• (c) This limit does not exist, since \( \sin(1/x) \) oscillates infinitely often between \(-1\) and \(1\) as \(x \to 0\) and the limit of the denominator \(x\) is \(0\).
(d) If $x < 3$, then $|x - 3| = 3 - x$, so

$$\lim_{x \to 3^-} \frac{x^2 - 2x - 3}{|x - 3|} = \lim_{x \to 3^-} \frac{x^2 - 2x - 3}{3 - x}$$

$$= \lim_{x \to 3^-} \frac{(x - 3)(x + 1)}{3 - x}$$

$$= -\lim_{x \to 3^-} (x + 1)$$

$$= -4$$
3. [15%] Evaluate the following limits involving infinity.

(a) \[ \lim_{x \to 1} \frac{x + 1}{(x - 1)^2(x - 2)} \]

(b) \[ \lim_{x \to \infty} \sqrt{\frac{x^2 + 2x + 4}{4x^2 + 2x + 1}} \]

(c) \[ \lim_{x \to -\infty} xe^{1/x} \]

Solution.

• (a) Since \((x - 1)^2 > 0\), we have
\[
\lim_{x \to 1} \frac{x + 1}{(x - 1)^2(x - 2)} = \lim_{x \to 1} \frac{x + 1}{x - 2} \cdot \lim_{x \to 1} \frac{1}{(x - 1)^2} = -2 \lim_{x \to 1} \frac{1}{(x - 1)^2} = -\infty.
\]

• (b) We have
\[
\lim_{x \to \infty} \sqrt{\frac{x^2 + 2x + 4}{4x^2 + 2x + 1}} = \lim_{x \to \infty} \sqrt{\frac{1 + 2/x + 4/x^2}{4 + 2/x + 1/x^2}} = \frac{1}{2}.
\]

• (c) We have
\[
\lim_{x \to -\infty} e^{1/x} = \lim_{t \to 0^{-}} e^t = e^{0} = 1,
\]
so
\[
\lim_{x \to -\infty} xe^{1/x} = \lim_{x \to -\infty} x = -\infty.
\]
4. [20%] Define a function \( f(x) \) for all real numbers except \( x = -1, 0, 2 \) by

\[
f(x) = \begin{cases} 
  \frac{1}{x^2} & \text{if } x < -1, \\
  -\frac{1}{x^3} & \text{if } -1 < x < 0, \\
  \frac{1}{x} & \text{if } 0 < x < 2, \\
  \frac{1}{x^2} & \text{if } x > 2.
\end{cases}
\]

(a) Evaluate \( \lim_{x \to -1^-} f(x) \) and \( \lim_{x \to -1^+} f(x) \). Can you choose a value of \( f(-1) \) so that \( f(x) \) is continuous at \( x = -1 \); if so, what is \( f(-1) \)?

(b) Evaluate \( \lim_{x \to 0^-} f(x) \) and \( \lim_{x \to 0^+} f(x) \). Can you choose a value of \( f(0) \) so that \( f(x) \) is continuous at \( x = 0 \); if so, what is \( f(0) \)?

(c) Evaluate \( \lim_{x \to 2^-} f(x) \) and \( \lim_{x \to 2^+} f(x) \). Can you choose a value of \( f(2) \) so that \( f(x) \) is continuous at \( x = 2 \); if so, what is \( f(2) \)?

Solution.

- (a) We have

\[
\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} \frac{1}{x^2} = 1,
\]

\[
\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} -\frac{1}{x^3} = 1.
\]

Since the left and right limits are the same, \( \lim_{x \to -1} f(x) = 1 \) exists and \( f(x) \) has a removable discontinuity at \( x = -1 \). We can make \( f(x) \) continuous at \( -1 \) by defining \( f(-1) = 1 \).

- (b) We have

\[
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} -\frac{1}{x^3} = \infty,
\]

\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{1}{x} = \infty,
\]

so \( f(x) \) has an infinite discontinuity at \( x = 0 \) and \( \lim_{x \to 0} f(x) \) does not exist. We cannot make \( f(x) \) continuous at \( 0 \) by any choice of the value of \( f(0) \).
(c) We have

\[
\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{1}{x} = \frac{1}{2},
\]

\[
\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{1}{x^2} = \frac{1}{4},
\]

so \( f(x) \) has a jump discontinuity at \( x = 2 \) and \( \lim_{x \to 2} f(x) \) does not exist. We cannot make \( f(x) \) continuous at 2 by any choice of the value of \( f(2) \).
5. [15%] (a) Suppose that a function $f(x)$ is defined for all $x$ in an interval about $c$, except possibly at $c$ itself. Give the precise $\varepsilon$-$\delta$ definition of

$$\lim_{x \to c} f(x) = L.$$ 

(b) Use the $\varepsilon$-$\delta$ definition to prove that

$$\lim_{x \to 2} (3x - 1) = 5.$$ 

Solution.

- (a) $\lim_{x \to c} f(x) = L$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \text{ implies that } |f(x) - L| < \varepsilon.$$ 

- (b) Let $\varepsilon > 0$ be given and choose

$$\delta = \frac{\varepsilon}{3} > 0.$$ 

Then $0 < |x - 2| < \delta$ implies that

$$|(3x - 1) - 5| = 3|x - 2|$$

$$< 3\delta$$

$$< \varepsilon,$$

which proves the result.