1. [20%] Compute the derivatives of the following functions. You do not need to simplify your answers.

(a) $7 \tan(\pi x) + 3x \tan^{-1}(5x)$  
(b) $\frac{x + x^3 + x^5}{1 + x^2 + x^4}$  
(c) $\sqrt{\sin(x^2)}$  
(d) $xe^{-2x} \ln(3x)$

Solution.

• (a)

$$\frac{d}{dx} \left[ 7 \tan(\pi x) + 3x \tan^{-1}(5x) \right] = 7 \cdot \sec^2(\pi x) \cdot \pi + 3 \tan^{-1}(5x) + 3x \cdot \frac{1}{1 + (5x)^2} \cdot 5$$

• (b)

$$\frac{d}{dx} \left[ \frac{x + x^3 + x^5}{1 + x^2 + x^4} \right] = \frac{(1 + 3x^2 + 5x^4) (1 + x^2 + x^4) - (2x + 4x^3) (x + x^3 + x^5)}{(1 + x^2 + x^4)^2}$$

Alternatively (and unintentionally), note that

$$\frac{x + x^3 + x^5}{1 + x^2 + x^4} = x$$

so

$$\frac{d}{dx} \left[ \frac{x + x^3 + x^5}{1 + x^2 + x^4} \right] = 1.$$

• (c)

$$\frac{d}{dx} \left[ \sqrt{\sin(x^2)} \right] = \frac{1}{2\sqrt{\sin(x^2)}} \cdot \cos(x^2) \cdot 2x$$

• (d)

$$\frac{d}{dx} \left[ xe^{-2x} \ln(3x) \right] = 1 \cdot e^{-2x} \ln(3x) + x(-2e^{-2x}) \ln(3x) + xe^{-2x} \cdot \frac{1}{3x} \cdot 3$$
2. [25\%] (a) Find an equation for the tangent line to the curve

\[2 \left( x^3 + y^3 \right) = 9xy\]

at the point \((x, y) = (1, 2)\).

(b) At what point does the tangent line in (a) intersect the line \(y = x\)?

**Solution.**

- (a) Using implicit differentiation, we get

\[6x^2 + 6y^2 \frac{dy}{dx} = 9y + 9x \frac{dy}{dx}.\]

- Canceling a factor of 3 and evaluating the result at \((x, y) = (1, 2)\), we get

\[2 + 8 \frac{dy}{dx} = 6 + 3 \frac{dy}{dx},\]

which implies that

\[\frac{dy}{dx} = \frac{4}{5}.\]

- The equation of the tangent line at \((x, y) = (1, 2)\) is therefore

\[y - 2 = \frac{4}{5}(x - 1).\]  \hspace{1cm} (1)

- (b) When \(y = x\) in (1), we get

\[5x - 10 = 4x - 4\]

so \(x = 6\), and the lines intersect at \((x, y) = (6, 6)\).
3. [25%] A growing column of ice is in the shape of a perfect cylinder whose height is six times its radius. If the volume of the ice is increasing at a rate of 0.5 inches³/hour, find the rate at which the height is increasing when the height of the column is 4 inches.

HINT. The volume $V$ of a cylinder with height $h$ and radius $r$ is $V = \pi r^2 h$.

Solution.

- Let $h(t)$ be the height, $r(t)$ the radius, and $V(t)$ the volume of the cylinder at time $t$. Then $h = 6r$ and

  $$V = \pi r^2 h = \frac{1}{36} \pi h^3.$$ 

- Differentiating this equation with respect to $t$, we get

  $$\frac{dV}{dt} = \frac{1}{12} \pi h^2 \frac{dh}{dt}.$$ 

- When $dV/dt = 1/2$ and $h = 4$, we have

  $$\frac{1}{2} = \frac{4\pi}{3} \frac{dh}{dt},$$ 

  so the rate at which the height is increasing is

  $$\frac{dh}{dt} = \frac{3}{8\pi} \text{ inches/hour.}$$ 

4. [15%] (a) Use the chain rule and the formula for the derivative of the arcsine to compute the derivative of the function

\[ f(x) = \cos \left( \sin^{-1} x \right). \]

(b) Compute the derivative of the function

\[ g(x) = \sqrt{1 - x^2}. \]

(c) What is the relationship between your answers in (a) and (b)? Explain why this happens.

Solution.

• (a) Using the chain rule, the derivative of the arcsine, and the definition of the arcsine, we get

\[
\begin{align*}
  f'(x) &= -\sin(\sin^{-1} x) \frac{d}{dx} \sin^{-1} x \\
        &= -\sin(\sin^{-1} x) \frac{1}{\sqrt{1-x^2}} \\
        &= -\frac{x}{\sqrt{1-x^2}}.
\end{align*}
\]

• (b) Using the chain rule, we get

\[
\begin{align*}
  g'(x) &= \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) = -\frac{x}{\sqrt{1-x^2}}.
\end{align*}
\]

• (c) The answers in (a) and (b) are the same. This is because

\[ \cos(\sin^{-1} x) = \sqrt{1-x^2}, \]

so the original functions in (a) and (b) are the same.

Remark. To show the identity in (c), suppose that \(-1 \leq x \leq 1\) and let \(\theta = \sin^{-1} x\). Then \(\sin \theta = x\) and \(-\pi/2 \leq \theta \leq \pi/2\). Using the Pythagorean identity

\[ \cos^2 \theta + \sin^2 \theta = 1, \]

we get

\[ \cos(\sin^{-1} x) = \cos \theta = \sqrt{1-\sin^2 \theta} = \sqrt{1-x^2}, \]

where we take the positive square root because \(\cos \theta \geq 0\) for \(-\pi/2 \leq \theta \leq \pi/2\).
5. [15\%] Define a function \( f(x) \) by

\[
 f(x) = \begin{cases} 
    e^{-1/x^2} & \text{if } x \neq 0, \\
    0 & \text{if } x = 0. 
\end{cases}
\]

(a) Use differentiation rules to compute the derivative \( f'(x) \) when \( x \neq 0 \).
(b) Use the definition of the derivative to write down an expression for \( f'(0) \) as a limit.
(c) Is \( f(x) \) differentiable at \( x = 0 \)? If so, what is \( f'(0) \)?

**Hint.** You can assume the following limit: \( \lim_{x \to \infty} xe^{-x^2} = 0. \)

**Solution.**

- (a) If \( x \neq 0 \), then
  \[
  f'(x) = \frac{2}{x^3} e^{-1/x^2}.
  \]

- (b) The limit definition of the derivative gives
  \[
  f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{e^{-1/h^2} - 0}{h} = \lim_{h \to 0} \frac{e^{-1/h^2}}{h}.
  \]

- (c) Writing \( 1/h = t \) and using the limit given in the hint, we find that
  \[
  \lim_{h \to 0^+} \frac{e^{-1/h^2}}{h} = \lim_{t \to \infty} te^{-t^2} = 0.
  \]
  Similarly, writing \( 1/h = -t \), we find that
  \[
  \lim_{h \to 0^-} \frac{e^{-1/h^2}}{h} = -\lim_{t \to \infty} te^{-t^2} = 0.
  \]
  Since both the left and right limits exists and are equal, it follows that
  \[
  \lim_{h \to 0} \frac{e^{-1/h^2}}{h} = 0,
  \]
  so \( f(x) \) is differentiable at \( x = 0 \) and \( f'(0) = 0 \).