1. [20%] Say if the following statements are true or false. (For this question only, you don’t have to explain your answers.)

(a) If \( \lim_{x \to 0} f(x) = 7 \), then \( f(0) = 7 \).
(b) If \( \lim_{x \to 0} f(x) = 1 \), then \( f(x) > 0 \) for all nonzero \( x \) that are sufficiently close to 0.
(c) If \( f(x) \) is a function with domain \([0,1]\) and \( f(0) = -1 \), \( f(1) = 2 \), then \( f(x) = 0 \) for some \( x \) in \((0,1)\).
(d) If \( \lim_{x \to 0^+} f(x) = 7 \), then \( \lim_{x \to 0} f(x^2) = 7 \).

Solution.

• Explanations are included in these solutions, although the question doesn’t ask for them.

• (a) False. (The existence of the limit as \( x \to 0 \) does not imply that 0 is in the domain of the function. Moreover, even if 0 is in the domain of the function, the function may be discontinuous at 0, in which case the value of the function at 0 is not equal the limit.)

• (b) True. (The function values \( f(x) \) are arbitrarily close to 1 for all nonzero \( x \) sufficiently close to 0, so they must be positive. More precisely, taking \( \epsilon = 1/2 \) — say — in the definition of the limit, we get that there exists \( \delta > 0 \) such that \( |f(x) - 1| < 1/2 \) for all \( x \) such that \( 0 < |x| < \delta \), which implies that \( f(x) > 1/2 > 0 \).)

• (c) False. (For example, the statement is not true for the function

\[
f(x) = \begin{cases} 
-1 & \text{if } 0 \leq x \leq 1/2, \\
2 & \text{if } 1/2 < x \leq 1.
\end{cases}
\]

The statement would be true if \( f(x) \) is required to be continuous, by the intermediate value theorem.)

• (d) True. (Writing \( t = x^2 \), we have \( t \to 0^+ \) as \( x \to 0 \) since \( x^2 \to 0 \) as \( x \to 0 \) and \( x^2 > 0 \) for \( x \neq 0 \). It follows that \( \lim_{x \to 0} f(x^2) = \lim_{t \to 0^+} f(t) \).)
2. [30%] Evaluate the following limits or say if they do not exist:

(a) \( \lim_{x \to 2} \frac{x^2 - 4}{x^2 - x - 2}; \)
(b) \( \lim_{x \to 1} \ln \left[ e^x + \ln \left( 3 - \frac{\tan(2x)}{x} \right) \right]; \)
(c) \( \lim_{x \to 0^+} \frac{\sin(\sqrt{x})}{x}; \)
(d) \( \lim_{x \to \infty} \frac{1}{x - \sqrt{x^2 + x}}. \)

Solution.

- (a) We have
  \[
  \lim_{x \to 2} \frac{x^2 - 4}{x^2 - x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 2)}{(x - 2)(x + 1)} = \lim_{x \to 2} \frac{x + 2}{x + 1} = \frac{4}{3}.
  \]

- (b) As the problem is written, we get from the continuity of the logarithm, exponential, and tangent functions that
  \[
  \lim_{x \to 1} \ln \left[ e^x + \ln \left( 3 - \frac{\tan(2x)}{x} \right) \right] = \ln \left[ e + \ln (3 - \tan 2) \right].
  \]
  This result is valid since \( 3 - \tan 2 \approx 5.19 \) and \( e + \ln (3 - \tan 2) \approx 4.36 \) are both positive and in the domain where \( \ln \) is continuous.

- As the problem was meant to be written, we have
  \[
  \lim_{x \to 0} \ln \left[ e^x + \ln \left( 3 - \frac{\tan(2x)}{x} \right) \right] = \ln \left[ 1 + \ln \left( 3 - \lim_{x \to 0} \frac{\tan(2x)}{x} \right) \right].
  \]
  Using the limit
  \[
  \lim_{x \to 0} \frac{\tan(2x)}{x} = 2 \lim_{x \to 0} \frac{\tan(2x)}{2x} = 2,
  \]
  \[
  \lim_{x \to 0} \ln \left( 3 - \frac{\tan(2x)}{x} \right) = \ln 3.
  \]
and the fact that $\ln 1 = 0$, we then get that

$$\lim_{x \to 0} \ln \left[ e^x + \ln \left( 3 - \frac{\tan(2x)}{x} \right) \right] = 0.$$ 

• (c) Using the limit $\sin \theta / \theta \to 1$ as $\theta \to 0$, we get that

$$\lim_{x \to 0^+} \frac{\sin \sqrt{x}}{x} = \lim_{x \to 0^+} \left( \frac{1}{\sqrt{x}} \right) \lim_{x \to 0^+} \left( \frac{\sin \sqrt{x}}{\sqrt{x}} \right) = \lim_{x \to 0^+} \frac{1}{\sqrt{x}} = \infty,$$

so the limit does not exist.

• (d) Using the difference of two squares to remove the square-root in the denominator and simplifying the result, we get

$$\lim_{x \to \infty} \frac{1}{x - \sqrt{x^2 + x}} = \lim_{x \to \infty} \frac{x + \sqrt{x^2 + x}}{(x + \sqrt{x^2 + x})(x - \sqrt{x^2 + x})} = \lim_{x \to \infty} \frac{x + \sqrt{x^2 + x}}{x^2 - (x^2 + x)} = -\lim_{x \to \infty} \frac{x + \sqrt{x^2 + x}}{x} = -\lim_{x \to \infty} \left( 1 + \sqrt{1 + 1/x} \right) = -2.$$
3. [20%] Define a function \( f(x) \) with domain all real numbers \( x \) by
\[
f(x) = \begin{cases} 
0 & \text{if } x \leq 0, \\
\sin(\pi/x) & \text{if } 0 < x < 2, \\
x & \text{if } x \geq 2.
\end{cases}
\]

At what points is \( f(x) \) continuous and at what points is \( f(x) \) discontinuous? What kinds of discontinuity does \( f(x) \) have?

Solution.

• The function \( f(x) \) is continuous at \( x \neq 0, 2 \), since 0, \( x \), and \( \sin(\pi/x) \) are continuous functions for \( x \neq 0 \).

• For \( x = 0 \), we have
\[
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 0 = 0,
\]
and
\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \sin(\pi/x)
\]
does not exist, since \( \sin(\pi/x) \) oscillates infinitely often between \(-1\) and \(1\) as \( x \to 0^+ \). It follows that \( \lim_{x \to 0} f(x) \) does not exist, so \( f(x) \) is not continuous at 0, where it has an oscillatory discontinuity.

• For \( x = 2 \), we have
\[
\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \sin(\pi/x) = \sin(\pi/2) = 1,
\]
\[
\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x = 2.
\]
Since the left and right limits are different, \( \lim_{x \to 2} f(x) \) does not exist, and \( f(x) \) has a jump discontinuity at \( x = 2 \).
4. [15%] (a) Write an expression for the slope of the tangent line to the graph $y = x^3$ at $x = 1$.
(b) Find the slope of the tangent line in (a).

Solution.

- (a) The slope $m$ of the tangent line is the limit as $x \to 1$ of the slope of the chords between the points $(1, 1)$ and $(x, x^3)$ on the graph:

$$m = \lim_{x \to 1} \frac{x^3 - 1}{x - 1}$$

Equivalently, writing $x = 1 + h$, we have

$$m = \lim_{h \to 0} \frac{(1 + h)^3 - 1}{h}$$

- (b) Factoring the numerator and denominator in the first expression and simplifying the result, we get

$$m = \lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x^2 + x + 1)}{x - 1} = \lim_{x \to 1} x^2 + x + 1 = 3.$$

Alternatively, expanding the cube in the second expression, we get

$$m = \lim_{h \to 0} \frac{(1 + h)^3 - 1}{h} = \lim_{h \to 0} \frac{1 + 3h + 3h^2 + h^3 - 1}{h} = \lim_{h \to 0} 3 + 3h + h^2 = 3.$$
5. [15%] (a) Suppose that a function $f(x)$ is defined for all $x$ in an interval about $c$, except possibly at $c$ itself. Give the precise $\varepsilon$-$\delta$ definition of

$$\lim_{x \to c} f(x) = L.$$ 

(b) Use the $\varepsilon$-$\delta$ definition to prove that

$$\lim_{x \to 0} (3 - 7x^2) = 3.$$ 

Solution.

- (a) $\lim_{x \to c} f(x) = L$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |x - c| < \delta$$

implies that $|f(x) - L| < \varepsilon$.

- (b) Let $\varepsilon > 0$ be given and choose

$$\delta = \sqrt{\frac{\varepsilon}{7}} > 0.$$ 

Then $0 < |x| < \delta$ implies that

$$|(3 - 7x^2) - 3| = 7|x|^2 < 7\delta^2 < \varepsilon,$$

which proves the result.