1. [20%] Compute the derivatives of the following functions. You do not need to simplify your answer.

(a) \(3\sqrt{1 + \tan x} - 2\sqrt{\cos x}\)  
(b) \(\frac{xe^{2x}}{\ln x}\)  
(c) \(\tan^{-1}(x^2)\sin^2 x\)  
(d) \((\tan^{-1}x)^2\sin(x^2)\).

Solution.

• (a)  
\[
\frac{d}{dx} \left[3\sqrt{1 + \tan x} - 2\sqrt{\cos x}\right] = \frac{3}{2\sqrt{1 + \tan x}} \cdot \sec^2 x - \frac{2}{2\sqrt{\cos x}}(- \sin x)
\]

• (b)  
\[
\frac{d}{dx} \left[\frac{xe^{2x}}{\ln x}\right] = \frac{(e^{2x} + x \cdot 2e^{2x}) \ln x - (1/x) \cdot xe^{2x}}{\ln^2 x}
\]

• (c)  
\[
\frac{d}{dx} \left[\tan^{-1}(x^2)\sin^2 x\right] = \frac{1}{1+x^4} \cdot 2x \cdot \sin^2 x + \tan^{-1}(x^2) \cdot 2 \sin x \cdot \cos x
\]

• (d)  
\[
\frac{d}{dx} \left[(\tan^{-1}x)^2\sin(x^2)\right] = 2\tan^{-1}x \cdot \frac{1}{1+x^2} \cdot \sin(x^2) + (\tan^{-1}x)^2 \cdot \cos(x^2) \cdot 2x
\]
2. [15%] (a) State the definition of the derivative $f'(c)$ of a function $f(x)$ at $x = c$.

(b) Suppose that $f(x) = 1/\sqrt{x}$. Compute $f'(9)$ from the definition of the derivative. (No credit for using differentiation rules.)

Solution.

• (a)

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

• (b)

$$f'(9) = \lim_{h \to 0} \frac{1/\sqrt{9+h} - 1/\sqrt{9}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{9} - \sqrt{9+h}}{\sqrt{9} \cdot \sqrt{9+h} \cdot h}$$

$$= \lim_{h \to 0} \frac{(3 - \sqrt{9+h})(3 + \sqrt{9+h})}{3\sqrt{9+h} \cdot (3 + \sqrt{9+h}) \cdot h}$$

$$= \lim_{h \to 0} \frac{-h}{3\sqrt{9+h} \cdot (3 + \sqrt{9+h})}$$

$$= -\lim_{h \to 0} \frac{1}{3 \cdot 3 \cdot (3 + 3)}$$

$$= -\frac{1}{54}.$$

• This answer agrees with the power rule:

$$\frac{d}{dx}x^{-1/2} = -\frac{1}{2}x^{-3/2}$$

$$= -\frac{1}{54} \text{ at } x = 9.$$
3. [20%] At what point does the tangent line to the curve

\[ 2x^2 + y^3 = x^3 + y^2 \]

at \((x, y) = (2, 1)\) intersect the \(x\)-axis?

**Solution.**

- Differentiating the equation of the curve with respect to \(x\), we get that

\[ 4x + 3y^2 \frac{dy}{dx} = 3x^2 + 2y \frac{dy}{dx}. \]

- Evaluation of this equation at \((x, y) = (2, 1)\) gives

\[ 8 + 3 \frac{dy}{dx} = 12 + 2 \frac{dy}{dx}, \]

so the slope of the tangent line is

\[ \frac{dy}{dx} = 4. \]

- The equation of the tangent line is

\[ y - 1 = 4(x - 2). \]

- The line intersect the \(x\)-axis when \(y = 0\), so \(x = 7/4\) and the point of intersection is \((x, y) = (7/4, 0)\).
4. [15%] Suppose that a particle moves a distance $s$ after time $t$ where

$$s = t - 2t^2 + t^3.$$ 

(a) At what times is the velocity of the particle equal to zero?

(b) How far has the particle moved when its acceleration is zero?

Solution.

- The velocity of the particle is

$$\frac{ds}{dt} = 1 - 4t + 3t^2 = (1 - 3t)(1 - t).$$

- The velocity is zero when $t = 1/3$ or $t = 1$.

- The acceleration of the particle is

$$\frac{d^2s}{dt^2} = -4 + 6t,$$

which is zero if $t = 2/3$. The location of the particle at that time is $s = 2/27$.

- To answer the question as stated, the particle moves forward from $s = 0$ to $s = 4/27$ for $0 \leq t \leq 1/3$, when its velocity is positive, then it moves backward from $s = 4/27$ to $s = 2/27$ for $1/3 \leq t \leq 2/3$, when its velocity is negative. The total distance moved by the particle is therefore

$$\frac{4}{27} + \frac{2}{27} = \frac{2}{9}.$$
5. [10%] Suppose that a function $f(x)$ has derivative $f'(x) = e^{x^2}$ and

$$g(x) = f(\sqrt{\sin x}).$$

Compute $g'(x)$

Solution.

- By the chain rule,

$$g'(x) = f'(\sqrt{\sin x}) \cdot \frac{1}{2\sqrt{\sin x}} \cdot \cos x.$$ 

- Using the expression for $f'(x)$ we then get that

$$g'(x) = e^{\sin x} \cdot \frac{1}{2\sqrt{\sin x}} \cdot \cos x.$$
6. [20%] A conical tank of height 3 m and radius 5 m at its top is filled with water at a rate of 0.2 m$^3$s$^{-1}$. Find the rate at which the height of the water is increasing when the height is 1 m.

**Hint.** The volume $V$ of a cone with height $h$ and radius $r$ is $V = \frac{1}{3} \pi r^2 h$.

**Solution.**

- Let $h(t)$ and $r(t)$ be the height of the water and the radius of the surface of the water at time $t$, respectively.

- By similarity $r/h = 5/3$, so $r = 5h/3$ and

$$V = \frac{25}{27} \pi h^3.$$

- Differentiating this equation with respect to $t$, we get

$$\frac{dV}{dt} = \frac{25}{9} \pi h^2 \frac{dh}{dt}.$$

- Evaluating this equation at $h = 1$ and $dV/dt = 1/5$ and solving for $dh/dt$, we get that

$$\frac{dh}{dt} = \frac{9}{125\pi} \text{ms}^{-1}.$$