1. Evaluate the following integrals:

(a) \( \int \frac{1}{y - y^3} \, dy \);  
(b) \( \int \frac{\tan^{-1} x}{x^2} \, dx \);  
(c) \( \int \frac{1}{\sqrt{1 + \sqrt{x}}} \, dx \).

Solution.

• (a) We use partial fractions:

\[
\frac{1}{y - y^3} = \frac{-1}{y^3(y - 1)(1 + y)} = \frac{A}{y} + \frac{B}{y - 1} + \frac{C}{y + 1}.
\]

Putting the partial fractions expansion over a common denominator, we get

\[
\frac{A}{y} + \frac{B}{y - 1} + \frac{C}{y + 1} = \frac{(A + B + C)y^2 + (B - C)y - A}{y(y - 1)(y + 1)}.
\]

• Hence, we require that

\[A + B + C = 0, \quad B - C = 0, \quad A = 1.\]

The solution is

\[A = 1, \quad B = -\frac{1}{2}, \quad C = -\frac{1}{2}.\]

• Thus,

\[
\int \frac{1}{y - y^3} \, dy = \int \left( \frac{1}{y} - \frac{1}{2(y - 1)} - \frac{1}{2(y + 1)} \right) \, dy
\]

\[= \ln |y| - \frac{1}{2}\ln |y - 1| - \frac{1}{2}\ln |y + 1| + C
\]

\[= \frac{1}{2} \ln \left| \frac{y^2}{y^2 - 1} \right| + C.
\]
• (b) To simplify the integral, we use integration by parts to differentiate
the inverse tangent. Setting
\[ u = \tan^{-1} x, \quad dv = \frac{1}{x^2} \, dx \]
\[ du = \frac{1}{1 + x^2} \, dx, \quad v = -\frac{1}{x} \]
in the formula for integration by parts, we get
\[ \int \frac{\tan^{-1} x}{x^2} \, dx = -\left( \frac{\tan^{-1} x}{x} \right) + \int \frac{1}{x(1 + x^2)} \, dx. \]

• The partial fractions expansion of the new integrand is
\[ \frac{1}{x(1 + x^2)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} \]
\[ = \frac{(A + B)x^2 + Cx + A}{x(x^2 + 1)} \]
Hence, we choose
\[ A + B = 0, \quad C = 0, \quad A = 1, \]
meaning that \( A = 1, B = -1, C = 0. \)

• Thus,
\[ \int \frac{1}{x(1 + x^2)} \, dx = \int \left( \frac{1}{x} - \frac{x}{x^2 + 1} \right) \, dx \]
\[ = \ln |x| - \frac{1}{2} \ln (x^2 + 1) + C. \]

• It follows that
\[ \int \frac{\tan^{-1} x}{x^2} \, dx = -\left( \frac{\tan^{-1} x}{x} \right) + \ln |x| - \frac{1}{2} \ln (x^2 + 1) + C. \]

• (c) We use the substitution \( u = \sqrt{x}, \) with
\[ du = \frac{1}{2\sqrt{x}} \, dx. \]
It follows that \( dx = 2u \, du \), so
\[
\int \frac{1}{\sqrt{1 + \sqrt{x}}} \, dx = \int \frac{2u}{\sqrt{1 + u}} \, du.
\]

• Next, we use the substitution
\[
v = \sqrt{1 + u},
\]
with
\[
dv = \frac{1}{2\sqrt{1 + u}} \, du.
\]
It follows that \( u = v^2 - 1 \) and \( du = 2v \, dv \), so
\[
\int \frac{2u}{\sqrt{1 + u}} \, du = \int \frac{2(v^2 - 1)}{v} \cdot 2v \, dv
\]
\[
= 4 \int (v^2 - 1) \, dv
\]
\[
= \frac{4}{3}v^3 - 4v + C.
\]

• Combining these results, and expressing the answer in terms of \( x \), we get
\[
\int \frac{1}{\sqrt{1 + \sqrt{x}}} \, dx = \frac{4}{3}v^3 - 4v + C
\]
\[
= \frac{4}{3}(1 + u)^{\frac{3}{2}} - 4(1 + u)^{\frac{1}{2}} + C
\]
\[
= \frac{4}{3}(1 + \sqrt{x})^{\frac{3}{2}} - 4(1 + \sqrt{x})^{\frac{1}{2}} + C.
\]

• We would get the same result if we combined the substitutions and set \( v = \sqrt{1 + \sqrt{x}} \) at the start.

**Remark.** The integrals can be evaluated by other (equally correct) methods. The answers, however, must always agree up to an additive constant.
2. Evaluate the following integrals:

(a) \[ \int x^2 \ln x \, dx; \quad (b) \int x \cos^3 (x^2) \sin (x^2) \, dx; \quad (c) \int \sqrt{36 - 4t^2} \, dt. \]

Solution.

• (a) We use integration by parts to differentiate the logarithm. Setting

\[
\begin{align*}
  u &= \ln x, & dv &= x^2 \, dx \\
  du &= \frac{1}{x} \, dx, & v &= \frac{1}{3} x^3
\end{align*}
\]

in the formula for integration by parts, we get

\[
\int x^2 \ln x \, dx = \frac{1}{3} x^3 \ln x - \frac{1}{3} \int x^2 \, dx
\]

\[
= \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 + C.
\]

• (b) First, we use the substitution \( u = x^2 \) with \( du = 2x \, dx \), which gives

\[
\int x \cos^3 (x^2) \sin (x^2) \, dx = \frac{1}{2} \int \cos^3 u \sin u \, du.
\]

• Next, we use the substitution \( v = \cos u \) with \( dv = -\sin u \, du \), which gives

\[
\frac{1}{2} \int \cos^3 u \sin u \, du = -\frac{1}{2} \int v^3 \, dv
\]

\[
= -\frac{1}{8} v^4 + C.
\]

• Expressing this answer in terms of \( x \), we get

\[
\int x \cos^3 (x^2) \sin (x^2) \, dx = -\frac{1}{8} \cos^4 (x^2) + C.
\]
• (c) Using the trigonometric substitution

\[ t = 3 \sin \theta, \quad dt = 3 \cos \theta \, d\theta, \]

we get

\[ \int \sqrt{36 - 4t^2} \, dt = 2 \int \sqrt{9 - t^2} \, dt = 18 \int \cos^2 \theta \, d\theta. \]

• Using the double angle formula, we have

\[ 18 \int \cos^2 \theta \, d\theta = 9 \int (1 + \cos 2\theta) \, d\theta = 9\theta + \frac{9}{2} \sin 2\theta + C. \]

• We have \( \theta = \sin^{-1}(t/3) \) and, by the double angle formula for sine,

\[ \sin 2\theta = 2 \sin \theta \cos \theta = 2 \sin \theta \sqrt{1 - \sin^2 \theta} = \frac{2t}{3} \sqrt{1 - \frac{t^2}{9}} = \frac{2t}{9} \sqrt{9 - t^2}. \]

• It follows that

\[ \int \sqrt{36 - 4t^2} \, dt = 9 \sin^{-1} \left( \frac{t}{3} \right) + t\sqrt{9 - t^2} + C. \]
3. Let $R$ be the region of the $(x, y)$-plane bounded by the curves 

$$y = \ln x, \quad y = 1, \quad x = e^2.$$ 

(a) Express the area of $R$ as an integral with respect to $x$.
(b) Express the area of $R$ as an integral with respect to $y$.
(c) Evaluate the integrals in (a) and (b). Check that you get the same answer.
(d) Find the volume of the solid obtained by revolving $R$ about the the $y$-axis.

**Solution.**

- (a) The region $R$ consists of the points $(x, y)$ such that 
  $$e \leq x \leq e^2, \quad 1 \leq y \leq \ln x.$$ 
  Thus, splitting $R$ into vertical rectangles with of height $(\ln x - 1)$ and width $dx$, we may express the area $A$ of $R$ as
  $$A = \int_e^{e^2} (\ln x - 1) \, dx.$$ 

(b) Since $y = \ln x$ implies that $x = e^y$, the region $R$ consists of the points $(x, y)$ such that 
  $$1 \leq y \leq 2, \quad e^y \leq x \leq e^2.$$ 
  Thus, splitting $R$ into horizontal rectangles with of width $(2 - e^y)$ and height $dy$, we may express the area $A$ of $R$ as
  $$A = \int_1^{e^2} (e^2 - e^y) \, dy.$$ 

- (c) For the integral in (a), using the standard integral of $\ln x$, we get
  $$A = \int_e^{e^2} (\ln x - 1) \, dx$$
  $$= \left[ x \ln x - x \right]_e^{e^2}$$
  $$= e^2 \ln (e^2) - 2e^2 - (e \ln e - 2e)$$
  $$= 2e^2 - 2e^2 - (e - 2e)$$
  $$= e.$$ 
  Alternatively, you can use integration by parts.
For the integral in (b), we get
\[
A = \int_{1}^{2} (e^2 - e^y) \, dy
\]
\[= \left[ e^2y - e^y \right]_{1}^{2}
\]
\[= 2e^2 - e^2 - (e^2 - e)
\]
\[= e.
\]
Thus, we get the same answer both ways.

- (d) We can compute the volume of the solid of revolution in two different ways, depending on how we split up $R$. Either method would be fine (though, as is often the case, the ‘washer’ method is simpler here).

- **Method 1: Shells.** The rotation of a vertical strip about the $y$-axis gives a ‘shell’ of height $h = \ln x - 1$, radius $r = x$, width $dx$, and volume
\[
dV = 2\pi rh \, dx.
\]
Hence, the volume $V$ of the resulting solid of revolution is given by
\[
V = 2\pi \int_{e}^{e^2} x (\ln x - 1) \, dx.
\]
We evaluate this integral by use of integration by parts, with
\[
u = \ln x - 1, \quad dv = x \, dx
\]
\[du = \frac{1}{x} \, dx, \quad v = \frac{1}{2} x^2,
\]
which gives
\[
V = 2\pi \left[ \frac{1}{2} x^2 (\ln x - 1) \right]_{e}^{e^2} - 2\pi \int_{e}^{e^2} \frac{1}{2} x^2 \cdot \frac{1}{x} \, dx
\]
\[= \pi \left[ e^4 (\ln e^2 - 1) - e^2 (\ln e - 1) \right] - \pi \int_{e}^{e^2} x \, dx
\]
\[= \pi e^4 - \pi \left[ \frac{1}{2} x^2 \right]_{e}^{e^2}
\]
\[= \frac{\pi}{2} \left( e^4 + e^2 \right).
\]
• Method 2: Washers. The rotation of a horizontal strip about the $y$-axis gives a ‘washer’ of inner radius $R = e^y$, outer radius $r = e^2$, height $dy$, and volume

$$dV = \pi \left( r^2 - R^2 \right) dy.$$ 

Hence, the volume $V$ of the solid of revolution is given by

$$V = \pi \left[ e^4y - \frac{1}{2} e^{2y} \right]^2_1$$

$$= \frac{\pi}{2} \left( e^4 + e^2 \right).$$
4. Express the length of the curve with parametric equation

\[ x = \frac{t}{1 + t^3}, \quad y = \frac{t^2}{1 + t^3} \quad (0 \leq t \leq 1) \]

as an integral with respect to \( t \). (You don’t need to evaluate the integral.)

Solution.

• The element of arclength is

\[
\begin{align*}
ds &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt.
\end{align*}
\]

• We have

\[
\begin{align*}
\frac{dx}{dt} &= \frac{1 \cdot (1 + t^3) - t \cdot 3t^2}{(1 + t^3)^2} \\
&= \frac{1 - 2t^3}{(1 + t^3)^2}, \\
\frac{dy}{dt} &= \frac{2t \cdot (1 + t^3) - t^2 \cdot 3t^2}{(1 + t^3)^2} \\
&= \frac{2t - t^4}{(1 + t^3)^2}.
\end{align*}
\]

• Hence

\[
\begin{align*}
\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= \frac{(1 - 2t^3)^2 + (2t - t^4)^2}{(1 + t^3)^4} \\
&= \frac{1 + 4t^2 - 4t^3 - 4t^5 + 4t^6 + t^8}{(1 + t^3)^4}
\end{align*}
\]

• The length \( L \) of the curve is given by

\[
L = \int_0^1 \sqrt{\frac{1 + 4t^2 - 4t^3 - 4t^5 + 4t^6 + t^8}{(1 + t^3)^4}} \, dt.
\]
5. The curve $y = x^3$, with $0 \leq x \leq 1$, is rotated about the $x$-axis.

(a) Express the surface area of the resulting surface of revolution as an integral with respect to $x$.

(b) Compute the surface area by evaluating this integral.

(c) Compute the volume of the corresponding solid of revolution.

Solution.

• (a) The surface area $A$ is given by

$$ A = \int 2\pi r \, ds $$

where $r = y$ is the radius of the surface of revolution, and $ds$ is the element of arclength on the curve.

• We have

$$ ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx $$

$$ = \sqrt{1 + (3x^2)^2} \, dx. $$

• The surface area is therefore given by the integral

$$ A = 2\pi \int_0^1 x^3 \sqrt{1 + 9x^4} \, dx. $$

• (b) Using the substitution $u = 1 + 9x^4$, with $du = 36x^3 \, dx$, we get

$$ A = \frac{2\pi}{36} \int_1^{10} \sqrt{u} \, du $$

$$ = \frac{\pi}{18} \left[ \frac{2}{3} u^{3/2} \right]_1^{10} $$

$$ = \frac{\pi}{27} \left( 10^{3/2} - 1 \right). $$
• (c) The volume $V$ of the solid of revolution about the $x$-axis is

\[
V = \int_0^1 \pi y^2 \, dx \\
= \int_0^1 \pi x^6 \, dx \\
= \pi \left[ \frac{1}{7} x^7 \right]_0^1 \\
= \frac{\pi}{7}.
\]
6. Let $[x]$ denote the floor function (that is, the greatest integer less than or equal to $x$ e.g. $[3/2] = 1$, $[14/5] = 2$). Is this function Riemann integrable on the interval $0 \leq x \leq 3$? If so, evaluate

$$\int_0^3 [x] \, dx.$$

Solution.

- The function is Riemann integrable, since it is piecewise continuous with a finite number of jump discontinuities (at $x = 0, 1, 2, 3$).

- Using the additive property of the integral and the fact that

$$[x] = \begin{cases} 
0 & \text{if } 0 \leq x < 1 \\
1 & \text{if } 1 \leq x < 2 \\
2 & \text{if } 2 \leq x < 3
\end{cases}$$

we get

$$\int_0^3 [x] \, dx = \int_0^1 [x] \, dx + \int_1^2 [x] \, dx + \int_2^3 [x] \, dx$$

$$= \int_0^1 0 \cdot dx + \int_1^2 1 \cdot dx + \int_2^3 2 \cdot dx$$

$$= 0 + 1 + 2 = 3.$$

- Note that the values of $[x]$ at the endpoints of these integrals do not affect them, since changing the values of a function at finitely many points does not change its integral.
7. Suppose that the velocity \( v(t) \) of a particle at time \( t \) is given by
\[
v(t) = t\sqrt{t+1}.
\]
(a) Compute the distance traveled by the particle for \( 0 \leq t \leq 3 \).
(b) What is the average velocity of the particle over that time interval?

Solution.

• (a) The distance traveled \( L \) is equal to the integral of the velocity with respect to time,
\[
L = \int_0^3 t\sqrt{t+1} \, dt.
\]

• Using the substitution \( u = \sqrt{t+1} \), with
\[
du = \frac{1}{2\sqrt{t+1}} \, dt, \quad t = u^2 - 1
\]
and \( dt = 2u \, du \), we get
\[
L = \int_1^2 (u^2 - 1) \cdot 2u \, du
\]
\[
= 2 \int_1^2 (u^4 - u^2) \, du
\]
\[
= 2 \left[ \frac{1}{5}u^5 - \frac{1}{3}u^3 \right]_1^2
\]
\[
= 2 \left[ \frac{32}{5} - \frac{8}{3} - \left( \frac{1}{5} - \frac{1}{3} \right) \right]
\]
\[
= \frac{116}{15}.
\]

• (b) The average velocity \( \bar{v} \) is given by
\[
\bar{v} = \frac{1}{3} \int_0^3 v(t) \, dt
\]
\[
= \frac{1}{3} \cdot \frac{116}{15}
\]
\[
= \frac{116}{45}.
\]
8. Suppose that an empty reservoir has the shape of a circular cone of radius \( a \) at its top and depth \( h \), where the top of the reservoir is at a height \( 2h \) above sea level and the bottom (the vertex of the cone) is at height \( h \). Calculate the work done against gravity required to fill the reservoir by pumping water from sea level into the reservoir. (Assume that the water only has to be pumped up to the current water level in the reservoir, not all the way up to the top.) Express your answer in terms of \( a \), \( h \), and the weight density \( w = \rho g \) of water.

Solution.

- Let \( z \) be the height measure from above the bottom of the reservoir. (It would be equally correct to use height \( y = z + h \) above sea level instead.)

- The reservoir cross section at height \( z \) is a circle of radius \( r = az/h \) (since the radius varies linearly from \( r = 0 \) at \( z = 0 \) to \( r = a \) at \( z = h \)).

- The volume \( dV \) of a ‘slice’ of the reservoir of thickness \( dz \) at height \( z \) is \( A(z) \, dz \) where \( A = \pi r^2 \) is the cross-sectional area. The mass \( dm \) of water required required to fill this ‘slice’ is \( \rho A(z) \, dz \), where \( \rho \) is the mass-density of water.

- Water at height \( z \) has to be pumped a vertical distance \( (z + h) \) from sea level. The work \( dW \) required to pump a mass \( dm \) of water is \( g(z + h) \, dm \), or

\[
dW = \rho g(z + h)A(z) \, dz = w(z + h)A(z) \, dz.
\]

- Thus, the total work \( W \) required is

\[
W = \int_0^h w(z + h)A(z) \, dz
\]

\[
= \int_0^h w(z + h)\pi \left( \frac{az}{h} \right)^2 \, dz
\]

\[
= \frac{\pi wa^2}{h^2} \int_0^h (z^3 + h z^2) \, dz
\]

\[
= \frac{\pi wa^2}{h^2} \left[ \frac{1}{4}z^4 + \frac{1}{3}hz^3 \right]_0^h
\]

\[
= \frac{7}{12} \pi wa^2 h^2.
\]
9. As a simplified model of the earth’s atmosphere, consider a stationary atmosphere in which the pressure \( p(z) \) and mass density \( \rho(z) \) vary with height \( 0 \leq z < \infty \). Let \( g \) be the gravitational acceleration (assumed constant).

(a) Explain briefly why

\[
p(z) = g \int_{z}^{\infty} \rho(s) \, ds.
\]

(You can assume that the improper integral converges at infinity.)

(b) Explain briefly why it follows that

\[
\frac{dp}{dz} = -g \rho(z).
\]

(c) Suppose that \( \rho(z) = cp(z) \) for some constant \( c \), meaning that the density is proportional to the pressure, and \( p(0) = p_0 \) where \( p_0 \) is the atmospheric pressure at the ground \( z = 0 \). Find \( p(z) \).

(d) At what height \( H \) is the pressure \( p(H) = p_0/e \) reduced by a factor \( e \) from its value at the ground?

Solution.

- (a) In hydrostatic equilibrium, the pressure \( p(z) \) at some height \( z \) in a fluid is equal to the gravitational force per unit horizontal area exerted by the fluid above. This force is equal to \( g \) times the mass per unit area of the fluid above, or

\[
p(z) = g \int_{z}^{\infty} \rho(s) \, ds.
\]

- (b) This equation follows from the fundamental theorem of calculus, and the fact that the integral changes sign when we exchange the limits:

\[
\frac{dp}{dz} = \frac{d}{dz} \int_{z}^{\infty} \rho(s) \, ds = -g \frac{d}{dz} \int_{z}^{\infty} \rho(s) \, ds = -g \rho(z).
\]

- (c) Using the formula \( \rho = cp \) to eliminate the density \( \rho \) from the equation in (b), and the condition at \( z = 0 \), we get

\[
\frac{dp}{dz} = -cgp, \quad p(0) = p_0.
\]
• The solution of this initial value problem is the exponential function

\[ p(z) = p_0 e^{-cz}. \]

• (d) The pressure decreases by a factor of \( e \) over a height

\[ H = \frac{1}{cg}. \]

**Remark.** The pressure \( p \), specific volume \( V \), and absolute temperature \( T \) of a unit mass of an ideal gas are related by the equation

\[ pV = RT, \]

where \( R \) is a gas constant. (For dry air, \( R = 287 \text{JKg}^{-1}\text{K}^{-1} \), where the absolute temperature \( T \) is measured in Kelvins K.) The density of the gas is given by \( \rho = 1/V \). Thus, \( \rho \) is proportional to \( p \) if the temperature \( T \) is constant, and in that case the constant of proportionality is given by

\[ c = \frac{1}{RT}. \]

In an isothermal atmosphere in hydrostatic equilibrium, the density and pressure decrease exponentially with height,

\[ p(z) = p_0 e^{z/H}, \quad \rho(z) = \rho_0 e^{z/H}. \]

The height \( H \) over which they decrease by a factor of \( e \) is called the *scale height* of the atmosphere, and it is given by

\[ H = \frac{RT}{g}. \]

Although the earth’s atmosphere is not exactly isothermal, its temperature in the first 70 km above the surface is approximately within 15% of 250 K. The corresponding scale height is \( H = 7.2 \text{ km} \).

Humans require an air pressure of at least 0.5 atmospheres to survive on a long-term basis. This sets an upper limit on habitable altitudes of about 5.8 km or 19,000 ft, e.g. for some communities in the Andes or Tibet. The peak of Mt. Everest, at an altitude of 8.85 km or 29,028 ft, is not habitable.
10. Define a function $F(x)$ by

$$F(x) = \int_0^{\sqrt{x}} e^{-t^2} \, dt.$$ 

Compute $F'(2)$.

**Solution.**

- Write $F$ as a composition,

  $$F(x) = G(\sqrt{x}),$$

  where

  $$G(y) = \int_0^y e^{-t^2} \, dt.$$ 

- By the fundamental theorem of calculus,

  $$G'(y) = e^{-y^2}.$$ 

- By the chain rule,

  $$F'(x) = G'(\sqrt{x}) \cdot \left(\sqrt{x}\right)' = G'(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} = \frac{e^{-x}}{2\sqrt{x}}.$$ 

- It follows that

  $$F'(2) = \frac{1}{2\sqrt{2}e^2}.$$
11. Suppose that the function $f(x)$ is defined on the interval $a \leq x \leq b$ with continuous first and second derivatives. If $f(a) = f(b) = 0$, show that

$$\int_a^b f(x)f''(x) \, dx \leq 0.$$  

HINT. Integrate by parts.

Solution.

- We use the integration by parts formula

$$\int_a^b u \, dv = [uv]_a^b - \int_a^b v \, du,$$

where $a, b$ are understood to be $x$-values, with

$$u = f, \quad dv = f'' \, dx$$

$$du = f' \, dx, \quad v = f'.$$

This gives

$$\int_a^b f(x) \cdot f''(x) \, dx = [f(x)f'(x)]_a^b - \int_a^b f'(x) \cdot f'(x) \, dx$$

- The boundary terms vanish because

$$[f(x)f'(x)]_a^b = f(b)f'(b) - f(a)f'(a) = 0 \cdot f'(b) - 0 \cdot f'(a) = 0.$$

- It follows that

$$\int_a^b f(x)f''(x) \, dx = -\int_a^b [f'(x)]^2 \, dx.$$

- Since a square is nonnegative and the integral of a nonnegative function is nonnegative, the right-hand side of this equation is less than or equal to zero, so

$$\int_a^b f(x)f''(x) \, dx \leq 0,$$
12. Show that
\[ 1 \leq \int_0^2 \frac{1}{1 + \sin^4(\pi x)} \, dx \leq 2. \]

Solution.

- Since \(|\sin \theta| \leq 1\) for any \(-\infty < \theta < \infty\), we have
  \[ 1 \leq 1 + \sin^4(\pi x) \leq 2. \]
  
  Therefore
  \[ \frac{1}{2} \leq \frac{1}{1 + \sin^4(\pi x)} \leq 1. \]

- It follows from the positivity properties of the integral that
  \[ \int_0^2 \frac{1}{2} \, dx \leq \int_0^2 \frac{1}{1 + \sin^4(\pi x)} \, dx \leq \int_0^2 1 \, dx, \]
  
  which implies that
  \[ 1 \leq \int_0^2 \frac{1}{1 + \sin^4(\pi x)} \, dx \leq 2. \]
13. (a) Use summation notation to write down the Riemann sum for
\[ \int_{0}^{2} x^5 \, dx \]
that is obtained by partitioning the interval \(0 \leq x \leq 2\) into \(n\) equal subintervals and evaluating the integrand at the left endpoint of each subinterval. Does this Riemann sum provide a lower bound or an upper bound of the integral, or neither?

(b) Write down an integral that corresponds to the following limit of Riemann sums:
\[ \lim_{n \to \infty} \left( \frac{2}{n} \sum_{k=1}^{n} \frac{1}{1 + 2k/n} \right). \]

Solution.

• (a) We partition the interval \(0 \leq x \leq 2\) into \(n\) subintervals of equal length
\[ \Delta x = \frac{2}{n} \]
with endpoints
\[ \{x_0, x_1, x_2, \ldots, x_k, \ldots, x_n\} \]
where
\[ x_k = \frac{2k}{n} \quad \text{for } k = 0, 1, 2, \ldots, n. \]

• The left endpoint of the \(k^{th}\)-interval, \(x_{k-1} \leq x \leq x_k\), is \(x_{k-1}\). Thus, taking \(c_k = x_{k-1}\) in the Riemann sum \(I_n = \sum_{k=1}^{n} f (c_k) \Delta x\), we get
\[
I_n = \sum_{k=1}^{n} (x_{k-1})^5 \Delta x
= \sum_{k=1}^{n} \left( \frac{2(k - 1)}{n} \right)^5 \frac{2}{n}
= \frac{2^6}{n^6} \sum_{k=1}^{n} (k - 1)^5
\]
• We can write this sum in other equivalent forms by changing the summation index; for example,

\[ I_n = \frac{2^6}{n^6} \sum_{j=0}^{n-1} j^5. \]

The \( j = 0 \) (or \( k = 1 \)) term could be omitted from the sum because it is 0.

• Since \( x^5 \) is an increasing function, its value at the left endpoint of a subinterval underestimates the function on the subinterval, and these Riemann sums provide a lower bound for the value of the integral.

• (b) Partitioning the interval \( 0 \leq x \leq 2 \) into \( n \) subintervals with equal length \( \Delta x = \frac{2}{n} \), and endpoints \( x_k = 2k/n \) for \( k = 0, 1, 2, \ldots, n \), we may write

\[ \frac{2}{n} \sum_{k=1}^{n} \frac{1}{1 + 2k/n} = \sum_{k=1}^{n} \frac{1}{1 + x_k} \Delta x. \]

These are Riemann sums (using the right endpoints of the subintervals) for the integral

\[ \int_0^2 \frac{1}{1 + x} \, dx = \lim_{n \to \infty} \left( \frac{2}{n} \sum_{k=1}^{n} \frac{1}{1 + 2k/n} \right). \]

**Remark 1.** The Riemann sums in (a) that use the right end point are

\[ J_n = \sum_{k=1}^{n} (x_k)^5 \Delta x = \frac{2^6}{n^6} \sum_{k=1}^{n} k^5. \]

Since \( x^5 \) is increasing, these Riemann sums provide an upper bound for the value of the integral.

**Remark 2.** The Riemann sums for \( x^5 \) can be evaluated exactly by use of the summation formula

\[ \sum_{k=1}^{n} k^5 = \frac{1}{12} n^2 (n + 1)^2 \left(2n^2 + 2n - 1\right). \]
This gives
\[ J_n = \frac{2^6}{12} \left[ \frac{n^2(n + 1)^2(2n^2 + 2n - 1)}{n^6} \right]. \]

Dividing the powers of \( n \) in the denominator into the numerator and taking the limit as \( n \to \infty \), we get
\[
\lim_{n \to \infty} J_n = \frac{2^6}{12} \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^2 \left( 2 + \frac{2}{n} - \frac{1}{n^2} \right) = \frac{2^6}{6}.
\]

The value of this limit (which defines the Riemann integral) agrees with the value of the integral computed by use of the fundamental theorem of calculus:
\[
\int_0^2 x^5 \, dx = \left[ \frac{1}{6} x^6 \right]_0^2 = \frac{2^6}{6}.
\]

Note that we obtain \( I_n \) by replacing \( n \) by \( n - 1 \) in the formula for \( J_n \), which gives
\[
I_n = \frac{2^6}{12} \left[ \frac{(n - 1)^2 n^2(2n^2 - 2n - 1)}{n^6} \right].
\]

The limit of the lower Riemann sums
\[
\lim_{n \to \infty} I_n = \frac{2^6}{12} \lim_{n \to \infty} \left( 1 - \frac{1}{n} \right)^2 \left( 2 - \frac{2}{n} - \frac{1}{n^2} \right) = \frac{2^6}{6}.
\]
is therefore the same as the limit of the upper Riemann sums, consistent with the fact that \( x^5 \) is Riemann integrable (because it is continuous).
14. (a) Show that the limit
\[
\lim_{\varepsilon \to 0^+} \left\{ \int_0^{\pi/2-\varepsilon} \tan x \, dx + \int_{\pi/2+\varepsilon}^{\pi} \tan x \, dx \right\}
\]
exists and evaluate it.

(b) Define the improper integral
\[
\int_0^{\pi} \tan x \, dx.
\]
Does it converge?

Solution.

- For \(0 < b < \pi/2\) and \(\pi/2 < a < \pi\), we consider the integrals
  \[
  I_b = \int_0^b \tan x \, dx, \quad J_a = \int_a^\pi \tan x \, dx.
  \]
  These Riemann integrals exist since \(\tan x\) is continuous on the intervals \([0, b]\) and \([a, \pi]\).

- Using the formula for the antiderivative of \(\tan x\),
  \[
  \int \tan x \, dx = -\ln |\cos x| + C,
  \]
  or writing \(\tan x = \sin x/\cos x\) and using logarithmic integration, we get
  \[
  I_b = [-\ln |\cos x|]_0^b = -\ln |\cos b|,
  J_a = [-\ln |\cos x|]_a^\pi = \ln |\cos a|.
  \]

- (a) From these equations, with \(b = \pi/2 - \varepsilon\) and \(a = \pi/2 + \varepsilon\), it follows that
  \[
  \int_0^{\pi/2-\varepsilon} \tan x \, dx + \int_{\pi/2+\varepsilon}^{\pi} \tan x \, dx
  = \ln \left| \cos \left(\frac{\pi}{2} + \varepsilon\right) \right| - \ln \left| \cos \left(\frac{\pi}{2} - \varepsilon\right) \right|
  = \ln \left| \frac{\cos (\pi/2 + \varepsilon)}{\cos (\pi/2 - \varepsilon)} \right|
  \]
• As $\varepsilon \to 0^+$, we have
  \[
  \cos \left( \frac{\pi}{2} - \varepsilon \right) \to 0, \quad \cos \left( \frac{\pi}{2} + \varepsilon \right) \to 0.
  \]

  Therefore, we may use l’Hôpital’s rule to find the limit of the fraction by differentiating the numerator and denominator with respect to $\varepsilon$. We get
  \[
  \lim_{\varepsilon \to 0^+} \frac{\cos \left( \frac{\pi}{2} + \varepsilon \right)}{\cos \left( \frac{\pi}{2} - \varepsilon \right)} = \lim_{\varepsilon \to 0^+} \frac{-\sin \left( \frac{\pi}{2} + \varepsilon \right)}{-\sin \left( \frac{\pi}{2} - \varepsilon \right)} = \lim_{\varepsilon \to 0^+} \frac{-\sin \left( \frac{\pi}{2} \right)}{-\sin \left( \frac{\pi}{2} \right)} = -1.
  \]

• Since the function $\ln |x|$ is continuous at $x = 1$, it follows that
  \[
  \lim_{\varepsilon \to 0^+} \ln \left| \frac{\cos \left( \frac{\pi}{2} + \varepsilon \right)}{\cos \left( \frac{\pi}{2} - \varepsilon \right)} \right| = \ln \left| \lim_{\varepsilon \to 0^+} \frac{\cos \left( \frac{\pi}{2} + \varepsilon \right)}{\cos \left( \frac{\pi}{2} - \varepsilon \right)} \right| = \ln | -1 | = 0.
  \]

Thus, the stated limit exists, and its value is
  \[
  \lim_{\varepsilon \to 0^+} \left\{ \int_0^{\pi/2 - \varepsilon} \tan x \, dx + \int_{\pi/2 + \varepsilon}^{\pi} \tan x \, dx \right\} = 0.
  \]

(b) The integrand $\tan x$ is discontinuous at $x = \pi/2$, and the improper integral is defined by
  \[
  \int_0^\pi \tan x \, dx = \lim_{b \to \pi/2^-} \int_0^b \tan x \, dx + \lim_{a \to \pi/2^+} \int_a^\pi \tan x \, dx.
  \]

• From above, we see that
  \[
  \int_0^b \tan x \, dx = -\ln |\cos b|.
  \]

This does not converge as $b \to \pi/2^-$, since $\cos b \to \cos(\pi/2) = 0$, and $-\ln |x|$ diverges to $\infty$ as $x \to 0$. Similarly,
  \[
  \int_0^{\pi/2} \tan x \, dx = \ln |\cos a|
  \]
  diverges to $-\infty$ as $a \to \pi/2^+$. 

Thus, the improper integral does not converge.

**Remark.** Note that the improper integrals

\[ \int_0^{\pi/2} \tan x \, dx, \quad \int_{\pi/2}^{\pi} \tan x \, dx \]

are required to exist separately. The limit

\[ \lim_{\varepsilon \to 0^+} \left\{ \int_0^{\pi/2-\varepsilon} \tan x \, dx + \int_{\pi/2+\varepsilon}^{\pi} \tan x \, dx \right\} \]

is finite because of a cancelation between the two terms when \( b = \pi/2 - \varepsilon, \ a = \pi/2 + \varepsilon \) are chosen symmetrically about the discontinuity at \( x = \pi/2 \). This is an example of a singular integral called a principal value integral, but it is not a convergent improper Riemann integral.
15. Solve the following initial value problem for $y(t)$:

$$\frac{dy}{dt} = ty^3, \quad y(0) = 1.$$ 

Sketch a graph of the solution. What happens to the solution?

Solution.

• Separating variables, we get

$$\int \frac{dy}{y^3} = \int t \, dt$$

Evaluation of the integrals gives

$$-\frac{1}{2y^2} = \frac{1}{2}t^2 + C$$

• The solution for $y$ is

$$y(t) = \frac{1}{\sqrt{C - t^2}}$$

(where $C$ is a different arbitrary constant from before).

• The initial condition $y(0) = 1$ implies that

$$1 = \frac{1}{\sqrt{C}}$$

or $C = 1$, so the solution is

$$y(t) = \frac{1}{\sqrt{1 - t^2}}.$$ 

• The solution goes to infinity as $t \to 1^-$ and $t \to -1^+$, so it is only defined for $-1 < t < 1$. 