5.1: 6

Use finite approximations to estimate the area under the graph of the function

\[ f(x) = x^3 \]

between \( x = 0 \) and \( x = 1 \) using

(a) a lower sum with two rectangles of equal width
(b) a lower sum with four rectangles of equal width,
(c) an upper sum with two rectangles of equal width,
(d) an upper sum with four rectangles of equal width.

(a) We divide the interval \([0, 1]\) into the two intervals \([0, \frac{1}{2}]\) and \([\frac{1}{2}, 1]\). Notice that this means the width of the rectangles, which your book represents by \( \Delta x \), is \( \frac{1}{2} \). Since this is a lower sum, for the interval \([0, \frac{1}{2}]\), we will be approximating \( f \) by the value \( f(0) = 0 \), and for the interval \([\frac{1}{2}, 1]\) we will be approximating \( f \) by the value \( f(\frac{1}{2}) = \frac{1}{8} \). Our estimate of the area is then:

\[ A \approx f(0) \Delta x + f(1/2) \Delta x = 0(\frac{1}{2}) + (1/8)(1/2) = 1/16. \]

Notice here that a lower sum uses the lowest value the function takes on in the interval. Since \( f \) is increasing in this instance, the lowest value is given by evaluation of \( f \) at the left endpoint of the interval. Nevertheless, do not confuse a ‘lower’ sum and a ‘left’ sum - see my solution for problem 5.1: 7 below for an instance where these are not the same thing.

(b) Since we’re using four rectangles now, we divide the interval \([0, 1]\) into the four intervals \([0, \frac{1}{4}]\), \([\frac{1}{4}, \frac{1}{2}]\), \([\frac{1}{2}, \frac{3}{4}]\), and \([\frac{3}{4}, 1]\). Here \( \Delta x = \frac{1}{4} \). Since this is a lower sum, we will be approximating \( f \) on \([0, \frac{1}{4}]\) by \( f(0) = 0 \), on \([\frac{1}{4}, \frac{1}{2}]\) by \( f(\frac{1}{4}) = \frac{1}{64} \), on \([\frac{1}{2}, \frac{3}{4}]\) by \( f(\frac{1}{2}) = \frac{1}{8} \), and on \([\frac{3}{4}, 1]\) by \( f(\frac{3}{4}) = \frac{27}{64} \). Our estimation of the area is then:

\[ A \approx \Delta x[f(0) + f(1/4) + f(1/2) + f(3/4)] = (1/4)(0 + 1/64 + 1/8 + 27/64) = 9/64. \]

(c) We again divide the interval \([0, 1]\) into the two intervals \([0, \frac{1}{2}]\) and \([\frac{1}{2}, 1]\), notice that this is unchanged from part (a). But now since this is an upper sum, for the interval \([0, \frac{1}{2}]\), we will be approximating \( f \) by the value \( f(1/2) = 1/8 \), and for the interval \([\frac{1}{2}, 1]\) we will be approximating \( f \) by the value \( f(1) = 1 \). Our estimate of the area is then:

\[ A \approx f(1) \Delta x + f(3) \Delta x = (1/8)(1/2) + (1)(1/2) = 9/16. \]

(d) Just as in part (c), we’re using four rectangles, so we divide the interval \([0, 1]\) into the four intervals \([0, 1/4]\), \([1/4, 1/2]\), \([1/2, 3/4]\), and \([3/4, 1]\). Since this is an upper sum, we will be approximating \( f \) on \([0, 1/4]\) by \( f(1/4) = 1/64 \), on \([1/4, 1/2]\) by \( f(1/2) = 1/8 \), on \([1/2, 3/4]\) by \( f(3/4) = 9/16 \), and on \([3/4, 1]\) by \( f(1) = 1 \). Our estimation of the area is then:

\[ A \approx \Delta x[f(1) + f(2) + f(3) + f(4)] = (1)(1/64 + 1/8 + 27/64 + 1) = 25/64. \]
5.1: 7

Note: this problem wasn’t actually assigned, but I accidently typed the solution up so I might as well leave it here.

Use finite approximations to estimate the area under the graph of the function

\[ f(x) = \frac{1}{x} \]

between \( x = 1 \) and \( x = 5 \) using

(a) a lower sum with two rectangles of equal width
(b) a lower sum with four rectangles of equal width,
(c) an upper sum with two rectangles of equal width,
(d) an upper sum with four rectangles of equal width.

(a) We divide the interval \([1, 5]\) into the two intervals \([1, 3]\) and \([3, 5]\). Notice that this means the width of the rectangles, which your book represents by \( \Delta x \), is 2. Since this is a lower sum, for the interval \([1, 3]\), we will be approximating \( f \) by the value \( f(1) = 1/3 \), and for the interval \([3, 5]\) we will be approximating \( f \) by the value \( f(3) = 1/5 \). Our estimate of the area is then:

\[ A \approx f(3) \Delta x + f(5) \Delta x = (1/3)(2) + (1/5)(2) = 16/15. \]

Notice here that a lower sum uses the lowest value the function takes on in the interval. Since \( f \) is decreasing in this instance, the lowest value is given by evaluation of \( f \) at the right endpoint of the interval.

(b) Since we’re using four rectangles now, we divide the interval \([1, 5]\) into the four intervals \([1, 2]\), \([2, 3]\), \([3, 4]\), and \([4, 5]\). Here \( \Delta x = 1 \). Since this is a lower sum, we will be approximating \( f \) on \([1, 2]\) by \( f(2) = 1/2 \), on \([2, 3]\) by \( f(3) = 1/3 \), on \([3, 4]\) by \( f(4) = 1/4 \), and on \([4, 5]\) by \( f(5) = 1/5 \). Our estimation of the area is then:

\[ A \approx \Delta x[f(2) + f(3) + f(4) + f(5)] = (1)(1/2 + 1/3 + 1/4 + 1/5) = 77/60. \]

(c) We again divide the interval \([1, 5]\) into the two intervals \([1, 3]\) and \([3, 5]\), notice that this is unchanged from part (a). But now since this is an upper sum, for the interval \([1, 3]\), we will be approximating \( f \) by the value \( f(1) = 1 \), and for the interval \([3, 5]\) we will be approximating \( f \) by the value \( f(3) = 1/3 \). Our estimate of the area is then:

\[ A \approx f(1) \Delta x + f(3) \Delta x = (1)(2) + (1/3)(2) = 8/3. \]

(d) Just as in part (c), we’re using four rectangles, so we divide the interval \([1, 5]\) into the four intervals \([1, 2]\), \([2, 3]\), \([3, 4]\), and \([4, 5]\). Since this is an upper sum, we will be approximating \( f \) on \([1, 2]\) by \( f(1) = 1 \), on \([2, 3]\) by \( f(2) = 1/2 \), on \([3, 4]\) by \( f(3) = 1/3 \), and on \([4, 5]\) by \( f(4) = 1/4 \). Our estimation of the area is then:

\[ A \approx \Delta x[f(1) + f(2) + f(3) + f(4)] = (1)(1 + 1/2 + 1/3 + 1/4) = 25/12. \]
(a) Inscribe a regular $n$-sided polygon inside a circle of radius 1 and compute the area of one of the $n$ congruent triangles formed by drawing radii to the vertices of the polygon.

(b) Compute the limit of the area of the inscribed polygon as $n \to \infty$.

(c) Repeat the computations in parts (a) and (b) for a circle of radius $r$.

(a) Notice that each isosceles triangle can be broken up into two right triangles. Once you’ve done that, things should be much easier to compute. The hypotenuse will of course be 1, the radius of the circle. One of the angles in the right triangle (the one at the center of the circle) will be $\pi/n$, and so the two sides of the triangle will be $\cos(\pi/n)$ and $\sin(\pi/n)$. Hence the area of each right triangle is $\frac{1}{2}bh = \frac{1}{2} \cos(\pi/n) \sin(\pi/n)$. The area of each isosceles triangle is twice the area of the right triangle, so it’s $A = \cos(\pi/n) \sin(\pi/n)$.

(b) The area of the inscribed $n$-gon is $n \sin(\pi/n) \cos(\pi/n)$. So to get the area of the inscribed polygon as $n \to \infty$, we simply have to take the limit:

$$\lim_{n \to \infty} n \sin(\pi/n) \cos(\pi/n).$$

Unfortunately, this is an indeterminate form, so we are forced to use L’Hôpital’s rule. Let’s make our life simpler before we do though. Certainly $\cos(\pi/n)$ has a nice finite limit: 1. So if we can show that $\lim_{n \to \infty} n \sin(\pi/n)$ is well defined and finite, we’ll have saved ourselves a lot of work. Let’s compute:

$$\lim_{n \to \infty} n \sin(\pi/n) = \lim_{n \to \infty} \frac{\sin(\pi/n)}{1/n} = \lim_{n \to \infty} \frac{\cos(\pi/n) \frac{\pi}{n^2}}{-\frac{1}{n^2}} = \lim_{n \to \infty} \pi \cos(\pi/n) = \pi.$$

In case you haven’t used L’Hôpital’s rule for a while, you should go back and brush up on it, and make sure you understand the steps that I’ve done in the above computation. The first step is simply putting the equation in a $0/0$ or $\infty/\infty$ form, the second step is where I actually differentiate and apply L’Hôpital’s rule. Now since $\lim_{n \to \infty} n \sin(\pi/n)$ exists and is finite, we have that:

$$\lim_{n \to \infty} n \sin(\pi/n) \cos(\pi/n) = \left( \lim_{n \to \infty} n \sin(\pi/n) \right) \left( \lim_{n \to \infty} \cos(\pi/n) \right) = \pi.$$

This is of course what we expect, since the area of a circle of radius 1 is $\pi$.

(c) The only difference here from part (a) is that the hypotenuse of our right triangle will now be $r$, instead of 1, so when we do our trig we will find that the two sides of the triangle are $r \sin(\pi/n)$ and $r \cos(\pi/n)$. If you trace everything through, then the area of the $n$-gon will be $r^2 n \sin(\pi/n) \cos(\pi/n)$, and so then taking the limit as $n \to \infty$ gives:

$$\lim_{n \to \infty} r^2 n \sin(\pi/n) \cos(\pi/n) = r^2 \lim_{n \to \infty} 2n \sin(\pi/n) \cos(\pi/n) = \pi r^2,$$

which is of course the area of a circle of radius $r$.

Notice that there are probably about as many ways to do this problem as there are trig identities.
5.2: 16

Write $-1/5 + 2/5 - 3/5 + 4/5 - 5/5$ in sigma notation.

Notice that:

$$\frac{-1}{5} + \frac{2}{5} - \frac{3}{5} + \frac{4}{5} - \frac{5}{5} = \frac{1}{5} (-1 + 2 - 3 + 4 - 5)$$

If we were writing $1 + 2 + 3 + 4 + 5$, that’d be easy, its given by $\sum_{k=1}^{5} k$. But we have an alternating sign, so we have to include that. We can do this with something that looks like $(-1)^k$, but we have to remember to check that we have the sign correct. Here we’ll get:

$$-1 + 2 - 3 + 4 - 5 = \sum_{k=1}^{5} (-1)^k k.$$

Now we just have to remember that factor of $1/5$, so our final answer should be:

$$\frac{-1}{5} + \frac{2}{5} - \frac{3}{5} + \frac{4}{5} - \frac{5}{5} = \frac{1}{5} \sum_{k=1}^{5} (-1)^k k.$$
5.2: 45

For $f(x) = 2x^3$ on the interval $[0, 1]$, find a formula for the Riemann sum obtained by dividing the interval $[0, 1]$ into $n$ equal subintervals and using the right-hand endpoint for each $c_k$. Then take the limit of these sums as $n \to \infty$ to calculate the area under the curve over $[0, 1]$.

For say, $n = 2$, our estimate would be $A_2 = (1/2)(f(1/2) + f(1)) = \frac{1}{2} \sum_{i=1}^{2} f(i/2)$. If you write out the formula for a few more values of $n$, you should be able to see that this generalize this to:

$$A_n = \frac{1}{n} \sum_{i=1}^{n} f(i/n),$$

where $A_n$ is the estimate of the area under the curve using $n$ equal subintervals. If we plug in our particular choice of $f$, we find:

$$A_n = \frac{1}{n} \sum_{i=1}^{n} 2(i/n)^3 = \frac{2}{n^4} \sum_{i=1}^{n} i^3.$$

We can evaluate this last sum using the formula from the book that $\sum_{i=1}^{n} i^3 = \frac{1}{4} n^2 (n+1)^2$. Plugging this in, we get:

$$A_n = \frac{2}{n^4} \sum_{i=1}^{n} i^3 = \frac{2}{n^4} \frac{1}{4} n^2 (n+1)^2 = \frac{1}{2} \frac{(n+1)^2}{n^2}.$$

To find the exact area under the curve, we just take the limit of $A_n$ as $n \to \infty$:

$$A = \lim_{n \to \infty} A_n = \lim_{n \to \infty} \frac{1}{2} \frac{(n+1)^2}{n^2} = \frac{1}{2}.$$

If you’re having trouble seeing that the limit of $\frac{(n+1)^2}{n^2}$ as $n \to \infty$ is 1, multiply both the numerator and denominator by $\frac{1}{n^2}$ and simplify.