5.3: 8

Express as a definite integral:

\[
\lim_{\|P\| \to 0} \sum_{k=1}^{n} (\tan c_k) \Delta x_k,
\]

Where \( P \) is a partition of \([0, \pi/4]\).

\[
\lim_{\|P\| \to 0} \sum_{k=1}^{n} (\tan c_k) \Delta x_k = \int_{0}^{\pi/4} \tan x \, dx.
\]

5.3: 36

Use the results of Equations (1) and (3) to evaluate \( \int_{\pi/2}^{0} \theta^2 \, d\theta \).

Here \( \theta \) - the variable of integration - is a ‘dummy’ variable. So if it makes it easier for you to apply equation (3), note that:

\[
\int_{\pi/2}^{0} \theta^2 \, d\theta = \int_{0}^{\pi/2} x^2 \, dx.
\]

By equation (3), we have that:

\[
\int_{\pi/2}^{0} \theta^2 \, d\theta = \frac{(\pi/2)^3}{3} - \frac{0^3}{3} = \pi^3/24.
\]

5.3: 79

Use the inequality \( \sin x \leq x \), which holds for \( x \geq 0 \), to find an upper bound for the value of \( \int_{0}^{1} \sin x \, dx \).

By the domination rule (pg 317), we have that:

\[
\int_{0}^{1} \sin x \, dx \leq \int_{0}^{1} x \, dx.
\]

From equation (1) on pg 320, we have that \( \int_{0}^{1} x \, dx = 1^2/2 - 0^2/2 = 1/2 \), hence:

\[
\int_{0}^{1} \sin x \, dx \leq \frac{1}{2}.
\]

In fact, the solution is \( 1 - \cos(1) \approx 0.46 \).
Guess an antiderivative for the integrand function. Validate your guess by differentiation and then evaluate the given definite integral.

Remember that \( \frac{d}{dx} \ln x = \frac{1}{x} \). With that in mind, something like \((\ln x)^2\) seems like a pretty good guess, though if you try that, you’ll quickly realize it’s not exactly right, but it’s not far off. The correct guess to make is \( \frac{1}{2}(\ln x)^2 \). Of course, this is only one possibility for the antiderivative - but there’s no reason to add a \(+C\) here since we’re evaluating a definite integral. Let’s check that my solution is correct:

\[
\frac{d}{dx} \frac{1}{2}(\ln x)^2 = \frac{1}{2} 2 \ln x \frac{d}{dx} \ln x = \frac{\ln x}{x}.
\]

So indeed it is. The definite integral can be evaluated as follows:

\[
\int_1^2 \frac{\ln x}{x} \, dx = \frac{1}{2} (\ln x)^2 \bigg|_1^2 = \frac{1}{2} [(\ln 2)^2 - (\ln 1)^2] = \frac{1}{2} (\ln 2)^2.
\]

The height \( H \) (ft) of a palm tree after growing for \( t \) years is given by

\[
H = \sqrt{t+1} + 5t^{1/3} \quad \text{for } 0 \leq t \leq 8.
\]

(a) Find the tree’s height when \( t = 0 \), \( t = 4 \), and \( t = 8 \).

(b) Find the tree’s average height for \( 0 \leq t \leq 8 \).

(a) We can simply evaluate \( H(t) \) at the given time values to find the height of the tree at that time. This gives:

\[
H(0) = \sqrt{0+1} + 5(0)^{1/3} = 1 \text{ ft},
\]

\[
H(4) = \sqrt{4+1} + 5(4)^{1/3} \approx 10.2 \text{ ft},
\]

\[
H(8) = \sqrt{8+1} + 5(8)^{1/3} = 13 \text{ ft}.
\]

(b) We’ll find the average value by using the formula:

\[
\text{Avg}(H) = \frac{1}{8-0} \int_0^8 H(t) \, dt.
\]

We then just compute. I’ll move the factor of 8 to the left side of the equation to make typing easier. I’ll exploit some of the properties of integrals to make this easier:

\[
8\text{Avg}(H) = \int_0^8 \sqrt{t+1} \, dt + 5 \int_0^8 t^{1/3} \, dt.
\]

At this point we’re just guessing antiderivatives. You can check for yourself that \( \frac{2}{3}(t+1)^{3/2} \) is an antiderivative for \( \sqrt{t+1} \), and that \( \frac{3}{4}t^{4/3} \) is an antiderivative for \( t^{1/3} \). Later we’ll learn how to calculate these directly. Using the fundamental theorem of calculus gives:

\[
8\text{Avg}(H) = \frac{2}{3}(t+1)^{3/2} \bigg|_0^8 + 5 \frac{3}{4}t^{4/3} \bigg|_0^8 = \frac{2}{3}(9)^{3/2} - \frac{2}{3}(1)^{3/2} + \frac{15}{4}(8)^{4/3} = \frac{232}{3}.
\]

Hence \( \text{Agv}(H) = \frac{1}{8} \frac{232}{3} = \frac{29}{3} \approx 9.7 \text{ feet} \).