4.8: 18

Find an antiderivative for each function. Check your answers by differentiation.

(a) $\sec x \tan x$

(b) $4 \sec 3x \tan 3x$

(c) $\sec \frac{\pi x}{2} \tan \frac{\pi x}{2}$

For this problem, it’s good to remember that $\frac{d}{dx} \sec x = \sec x \tan x$. From there, it’s just mentally undoing the chain rule. Here are the answers, feel free to check them for yourselves by differentiating.

(a) $\sec x$

(b) $\frac{4}{3} \sec 3x$

(c) $\frac{2}{\pi} \sec \frac{\pi x}{2}$

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Find the most general antiderivative or indefinite integral of:

$$\int \left( \frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}} \right) \, dx.$$ 

Check your answer by differentiation.

It’s useful often to express $\sqrt{\cdot}$ using a power to make antidifferentiation easier. I’ll also exploit the fact that constants can be pulled out of integrals, and that $\int (f + g) = \int f + \int g$ to simplify things and make guessing an antiderivative easier.

$$\int \left( \frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}} \right) \, dx = \frac{1}{2} \int x^{1/2} \, dx + 2 \int x^{-1/2} \, dx = \frac{1}{3} x^{3/2} + 4x^{1/2} + C. \quad (1)$$

You should differentiate this result to convince yourself it is valid.

5.5: 10

Compute:

$$\int \left( 1 - \cos \frac{t}{2} \right)^2 \sin \frac{t}{2} \, dt,$$

Using the substitution $u = 1 - \cos \frac{t}{2}$.

Since $u = 1 - \cos \frac{t}{2}$, we have that $du = \frac{1}{2} \sin \frac{t}{2} \, dt$. We rearrange this to find $\sin \frac{t}{2} = 2 \, du$, and then substitute to get:

$$\int \left( 1 - \cos \frac{t}{2} \right)^2 \sin \frac{t}{2} \, dt = 2 \int u^2 \, du = \frac{2}{3} u^3 + C = \frac{2}{3} \left( 1 - \cos \frac{t}{2} \right)^3 + C.$$
5.5: 42

Compute:
\[ \int \sqrt{\frac{x^4}{x^3 - 1}} \, dx. \]

Although this problem looks intimidating, it’s a good lesson in the fact that you should always simplify first:
\[ \int \sqrt{\frac{x^4}{x^3 - 1}} \, dx = \int x^2(x^3 - 1)^{-1/2} \, dx \]
At this point, it’s now easier to see that we should let \( u = x^3 - 1 \), so \( du = 3x^2 \, dx \). Rearranging that gives that:
\[ \int x^2(x^3 - 1)^{-1/2} \, dx = \frac{1}{3} \int u^{-1/2} \, du = \frac{2}{3} u^{1/2} + C = \frac{2}{3} \sqrt{x^3 - 1} + C. \]

5.5: 48

Compute:
\[ \int 3x^5 \sqrt{x^3 + 1} \, dx. \]

There’s no obvious simplification to be done. We take \( u = x^3 + 1 \), so \( du = 3x^2 \, dx \). Substitution then gives:
\[ \int 3x^5 \sqrt{x^3 + 1} \, dx = \int x^3 u^{1/2} \, du. \]

Uh oh! There’s still \( x \)'s in the equation, which is not good. But luckily, we can easily rearrange the relationship \( u = x^3 + 1 \) to find that \( x^3 = u - 1 \). Substituting this in then gives that:
\[ \int x^3 u^{1/2} \, du = \int (u-1)u^{1/2} \, du = \int \left( u^{3/2} - u^{1/2} \right) \, du = \frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} + C = \frac{2}{5} (x^3+1)^{5/2} - \frac{2}{3} (x^3+1)^{3/2} + C. \]
This is a nice example of an integral which you can find easily using the substitution method but would be very difficult to just ‘guess’.

5.5: 58

Compute:
\[ \int \frac{dx}{x \sqrt{x^4 - 1}}. \]

This one is not so obvious. If you let \( u = x^2 \), we have \( du = 2x \, dx \). Then upon substitution:
\[ \int \frac{dx}{x \sqrt{x^4 - 1}} = \frac{1}{2} \int \frac{2x \, dx}{x^2 \sqrt{u^2 - 1}} = \frac{1}{2} \int \frac{du}{u \sqrt{u^2 - 1}} = \frac{1}{2} \sec^{-1} u + C = \frac{1}{2} \sec^{-1}(x^2) + C. \]
At this point, evaluation of the integral \( \int \frac{du}{u \sqrt{u^2 - 1}} \) simply requires remembering the derivative of \( \sec^{-1} x \); which can be found on page 189 of your book. Later we’ll learn how to evaluate integrals like these more naturally.