6.1: 10

The base of the solid is the disk \( x^2 + y^2 \leq 1 \). The cross-sections by planes perpendicular to the \( y \)-axis between \( y = -1 \) and \( y = 1 \) are isosceles right triangles with one leg in the disk. (See picture in book).

First we need to determine the length of the chord that forms the bottom of the triangle as a function of \( y \).

Form a triangle by drawing a line from the midpoint of the chord to the center of the circle (length \( |y| \)), and then another line from the center of the circle to the end of the chord (length 1). Using the pythagorean theorem gives that the chord has a length of \( 2\sqrt{1 - y^2} \). Then the area of the triangle is given by:

\[
A(y) = \frac{1}{2}bh = \frac{1}{2} \left(2\sqrt{1 - y^2}\right) \left(2\sqrt{1 - y^2}\right) = 2(1 - y^2).
\]

The volume of the solid is then found by integrating:

\[
V = \int_{-1}^{1} A(y) \, dy = 2 \int_{0}^{1} (1 - y^2) \, dy = 4 \left( y - \frac{y^3}{3} \right) \bigg|_{0}^{1} = \frac{8}{3}.
\]

The second equality is due to the symmetry of the problem.

6.1: 18

Find the volume of the solid generated by revolving the shaded region (see book) about the \( x \)-axis.

The radius is \( \sin x \cos x \), so using the disk method:

\[
V = \int_{0}^{\pi/2} \pi (\sin x \cos x)^2 \, dx.
\]

There are probably quite a few ways to do this integral, but using the fact that \( 2 \sin x \cos x = \sin 2x \), and that \( \sin^2 x = \frac{1}{2}(1 - \cos 2x) \), we find:

\[
V = \frac{\pi}{8} \int_{0}^{\pi/2} (1 - \cos 4x) \, dx = \frac{\pi^2}{16} - \frac{1}{4} \int_{0}^{\pi} \cos x = \frac{\pi^2}{16}.
\]

In the last step I’ve just made the \( u \)-substitution \( u = 4x \) in my head; and used the fact that the average value of \( \cos x \) over its entire period is 0.

6.1: 40

Find the volume of the solid generated by revolving the region bounded by \( y = 2\sqrt{x} \), \( y = 2 \), and \( x = 0 \) about the \( x \)-axis.

Here’s its first necessary to sketch the region. Once you’ve done that, you should be able to pick off that the limits of integration in \( x \) are 0 and 1, and that the inner radius is \( r = 2\sqrt{x} \) and the outer radius is \( R = 2 \).

Then using the washer method, the volume is given by:

\[
V = \int_{0}^{1} \pi \left[ 2^2 - (2\sqrt{x})^2 \right] \, dx = \pi \left( 4 - \int_{0}^{1} 4x \, dx \right) = 4\pi \left( 1 - \frac{x^2}{2} \bigg|_{0}^{1} \right) = 2\pi.
\]
Use the shell method to find the volumes of the solids generated by revolving the region bounded by \( y = x^4 \) and \( y = 4 - 3x^2 \) about (a) the line \( x = 1 \) and (b) the \( x \)-axis.

First, sketch the region. For part (a), notice that the curves intersect at \((-1, 1)\) and \((1, 1)\). The radius will be given by \( 1 - x \), since we’re revolving about the line \( x = 1 \); and the height of the shell will be \((4 - 3x^2) - x^4\).

The setup for the shell method is then:

\[
V = \int_{-1}^{1} 2\pi (1 - x)(4 - 3x^2 - x^4) \, dx = \frac{56}{5}\pi.
\]

The integration is done by just expanding the product. Computing this volume would have been harder using the washer method, since it would involve splitting the region of integration in \( y \) into \([0, 1] \) and \([1, 4]\).

For part (b), the radius is just \( y \), and the limits of integration are \( y = 0 \) and \( y = 4 \); but the height changes - for \( y \in [0, 1] \), the height is given by \( 2x^4 \), which in terms of \( y \) is \( 2y^{1/4} \), but for \( y \in [1, 4] \), the height is given by \( 2(4 - x^4) \), which in terms of \( y \) is \( 2\sqrt{(4 - y)/3} \). Then the volume is given by:

\[
V = \int_{0}^{1} 2\pi y(2y^{1/4}) \, dy + \int_{1}^{4} 2\pi y(2\sqrt{(4 - y)/3}) \, dy = \frac{872}{45}\pi.
\]

A \( u \) substitution is required to compute the second integral. The computation of this volume would have been easier using the washer method.

Derive the equation for the volume of a sphere of radius \( r \) using the shell method.

We’ll imagine rotating about the \( y \)-axis. Then the radius is \( x \), the bounds of integration are from \( x = 0 \) to \( x = r \), and the height of the shell is \( 2\sqrt{r^2 - x^2} \). Then the volume is given by:

\[
V = \int_{0}^{r} 2\pi x \left(2\sqrt{r^2 - x^2}\right) \, dx = -2\pi \left(\frac{2}{3}\right) (r^2 - x^2)^{3/2} \bigg|_{0}^{r} = -\frac{4\pi}{3} (0 - r^3) = \frac{4}{3} \pi r^3.
\]