8.1: 8

Evaluate the following integral using integration by parts:

\[ \int xe^{3x} \, dx. \]

We choose \( u \) to be \( x \). Then we have:

\[
\begin{align*}
  u &= x & \quad dv &= e^{3x} \, dx \\
  du &= dx & \quad v &= \frac{1}{3} e^{3x}
\end{align*}
\]

Applying the integration by parts formula gives:

\[
\int xe^{3x} \, dx = \frac{1}{3} xe^{3x} - \int \frac{1}{3} e^{3x} \, dx = \frac{1}{3} xe^{3x} - \frac{1}{9} e^{3x} + C.
\]

8.1: 68

Use the formula

\[ \int f^{-1}(x) \, dx = xf^{-1}(x) - \int f(y) \, dy, \]  \hspace{1cm} (1)

where \( y = f^{-1}(x) \), to evaluate the integral:

\[ \int \tan^{-1}(x) \, dx. \]

Express your answer in terms of \( x \).

By the formula, we have:

\[
\int \tan^{-1}(x) \, dx = x \tan^{-1}(x) - \int \tan y \, dy
\]

\[
= x \tan^{-1}(x) - \ln(\cos y) + C
\]

\[
= x \tan^{-1}(x) - \ln(\cos(\tan^{-1} x)) + C
\]

\[
= x \tan^{-1}(x) + \frac{1}{2} \ln(1 + x^2) + C
\]

There are two steps in there that might require some explaining. To integrate \( \tan y \), rewrite \( \tan y \) as \( \sin y / \cos y \), and make a \( u \)-substitution. And to simplify \( \cos(\tan^{-1} x) \), let \( \theta = \tan^{-1} x \), and draw a reference triangle with the side opposite of \( \theta \) having length \( x \) and the side adjacent to \( \theta \) having length 1. Taking the \( \cos \) of \( \theta \) should then give you something similar to my result, though you’ll have to use \( \ln \) rules to manipulate it to the form I’ve given.
8.2: 24

Evaluate the integral:

\[ \int_0^\pi \sqrt{1 - \cos 2x} \, dx \]

To evaluate this integral we’ll need to use the trig identity:

\[ \sin^2 \theta = \frac{1 - \cos 2\theta}{2}. \]

Rearranging this gives something that looks quite similar to our integrand:

\[ \sqrt{2} \sin \theta = \sqrt{1 - \cos 2\theta}. \]

Plugging this in, we see that:

\[ \int_0^\pi \sqrt{1 - \cos 2x} \, dx = \sqrt{2} \int_0^\pi \sin x \, dx = -\sqrt{2} \cos \bigg|_0^\pi = 2\sqrt{2}. \]

8.2: 34

Evaluate the integral:

\[ \int \sec x \tan^2 x \, dx. \]

Sometimes a small change can make a big difference - look at problem 33, which is ever so slightly different but can be done with a simple \( u \)-substitution of \( u = \sec x \). This one is not so simple. One way to go is to integrate by parts. Let:

\[
\begin{align*}
  u &= \tan x & dv &= \sec x \tan x \, dx \\
  du &= \sec^2 x \, dx & v &= \sec x
\end{align*}
\]

Then applying the integration by parts formula, we find that:

\[ \int \sec x \tan^2 x \, dx = \sec x \tan x - \int \sec^3 x \, dx. \]

Here we use the fact that \( \sec^2 x = 1 + \tan^2 x \). Then:

\[ \int \sec x \tan^2 x \, dx = \sec x \tan x - \int \sec x \tan x \, dx - \int \sec x \tan^2 x \, dx. \]

Notice now that the final integral on the right is what we started with. We can move this back over to the left to get:

\[ 2 \int \sec x \tan^2 x \, dx = \sec x \tan x - \int \sec x \, dx. \]

All that remains is to integrate \( \sec x \). To do this, multiply by 1 as follows:

\[ \int \sec x \, dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} \, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx. \]

Notice now that the numerator is the derivative of the denominator, and so we can apply the \( u \)-substitution \( u = \sec x + \tan x \). Putting everything together gives:

\[ \int \sec x \tan^2 x \, dx = \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln (\sec x + \tan x). \]
8.3: 4

Evaluate the integral:
\[ \int_0^2 \frac{1}{8 + 2x^2} \, dx. \]

We first put this into a better form for working with:
\[ \int_0^2 \frac{1}{8 + 2x^2} \, dx = \frac{1}{8} \int_0^2 \frac{1}{1 + \left(\frac{x}{2}\right)^2} \, dx. \]

We now make the substitution \( \frac{x}{2} = \tan \theta \). Then \( dx = 2 \sec^2 \theta \, d\theta \). Plugging in (and not forgetting to change our limits!), we get:
\[ \frac{1}{8} \int_0^2 \frac{1}{1 + \left(\frac{x}{2}\right)^2} \, dx = \frac{1}{8} \int_0^{\tan^{-1}(1)} \frac{1}{1 + \tan^2 \theta} 2 \sec^2 \theta \, d\theta = \frac{1}{4} \int_0^{\pi/4} d\theta = \frac{\pi}{16}. \]

8.3: 38

Use an appropriate substitution and then a trigonometric substitution to evaluate the integral:
\[ \int_1^e \frac{1}{y \sqrt{1 + \ln(y)^2}} \, dy. \]

We begin by letting \( x = \ln y \), which gives \( dx = \frac{1}{y} \, dy \). Then substituting gives:
\[ \int_1^e \frac{1}{y \sqrt{1 + (\ln y)^2}} \, dy = \int_0^1 \frac{1}{\sqrt{1 + x^2}} \, dx. \]

We now let \( x = \tan \theta \), and \( dx = \sec^2 \theta \, d\theta \). Then substitution gives:
\[ \int_0^1 \frac{1}{\sqrt{1 + x^2}} \, dx = \int_0^{\pi/4} \frac{1}{\sqrt{1 + \tan^2 \theta}} \sec^2 \theta \, d\theta. \]

Simplifying gives an integral we’ve seen before in these solutions:
\[ \int_0^{\pi/4} \frac{1}{\sqrt{1 + \tan^2 \theta}} \sec^2 \theta \, d\theta = \int_0^{\pi/4} \sec \theta \, d\theta = \ln(\sec \theta + \tan \theta) \bigg|_0^{\pi/4} = \ln(1 + \sqrt{2}). \]