1. Do the following sequences \( \{a_n\} \) converge or diverge as \( n \to \infty \)? If a sequence converges, find its limit. Justify your answers.

   (a) \( a_n = \frac{2n^2 + 3n^3}{2n^3 + 3n^2} \);
   (b) \( a_n = \cos(n\pi) \);
   (c) \( a_n = \frac{\sin(n^2)}{n^2} \).

**Solution.**

- (a) Dividing the numerator and denominator by \( n^3 \), we have
  \[
  a_n = \frac{2/n + 3}{2 + 3/n} \to \frac{3}{2} \quad \text{as } n \to \infty
  \]
  so the sequence converges to \( 3/2 \).

- (b) We have
  \[
  a_n = \cos(n\pi) = (-1)^n
  \]
  so the sequence diverges since its terms oscillates between 1 and \(-1\).

- (c) We have
  \[
  |a_n| = \left| \frac{\sin(n^2)}{n^2} \right| \leq \frac{1}{n^2} \to 0 \quad \text{as } n \to \infty
  \]
  so the sequence converges to zero.
2. Do the following series converge or diverge? State clearly which test you use.

(a) \( \sum_{n=1}^{\infty} \frac{n + 4}{6n - 17} \)

(b) \( \sum_{n=2}^{\infty} \sqrt{\frac{n}{n^4 + 7}} \)

(c) \( \sum_{n=1}^{\infty} \frac{(-5)^{n+1}}{(2n)!} \)

(d) \( \sum_{n=3}^{\infty} \frac{\ln n}{n} \)

(e) \( \frac{1}{14} + \frac{1}{24} - \frac{1}{34} + \frac{1}{44} + \frac{1}{54} - \frac{1}{64} + \frac{1}{74} - \frac{1}{84} + \cdots \)

(f) \( \sum_{n=1}^{\infty} \left[ e^n - e^{n+1} \right] \)

Solution.

- We write each series as \( \sum c_n \).
  - (a) We have
    \[
    \lim_{n \to \infty} c_n = \lim_{n \to \infty} \frac{n + 4}{6n - 17} = \frac{1}{6}.
    \]
    Since this limit is nonzero, the series diverges by the \( n \)-th term test.
  - (b) We have
    \[
    0 \leq c_n = \sqrt{\frac{n}{n^4 + 7}} \leq \sqrt{\frac{n}{n^4}} = \frac{1}{n^{3/2}}.
    \]
    Therefore the series converges by the comparison test, since the \( p \)-series with \( p = 3/2 > 1 \) converges.
• (c) We have
\[
\frac{|c_{n+1}|}{c_n} = \left| \frac{(-5)^{n+2}/(2n+2)!}{(-5)^{n+1}/(2n)!} \right| = \frac{5(2n)!}{(2n+2)!} = \frac{5}{(2n+1)(2n+2)}.
\]
It follows that
\[
\lim_{n \to \infty} \frac{|c_{n+1}|}{c_n} = 0.
\]
Since this limit exists and is less than 1, the series converges absolutely by the ratio test, and therefore it converges.

• (d) Since \(\ln n > 1\) for \(n \geq 3\), we have
\[
c_n = \frac{\ln n}{n} > \frac{1}{n} \quad \text{for } n \geq 3.
\]
Since the harmonic series \(\sum 1/n\) diverges, the series diverges by the comparison test.

• (e) We have
\[
|c_n| = \frac{1}{n^4}
\]
so the series converges absolutely, since the \(p\)-series with \(p = 4\) converges. Therefore the series converges since any absolutely convergent series is convergent.

• (f) Either note that
\[
c_n = e^n - e^{n+1} = e^n (1 - e)
\]
diverges to \(-\infty\) as \(n \to \infty\), so the series diverges by the \(n\)th term test. Or note that the series is a telescoping series of the form \(c_n = b_n - b_{n+1}\) with \(b_n = e^n\) and the limit of \(b_n\) as \(n \to \infty\) does not exist, so the series is a divergent telescoping series.
3. Determine the interval of convergence (including the endpoints) for the following power series. State explicitly for what values of \( x \) the series converges absolutely, converges conditionally, and diverges. Specify the radius of convergence \( R \) and the center of the interval of convergence \( a \).

\[
\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n} (x - 1)^n.
\]

Solution.

- Applying the ratio test, we have

\[
\lim_{n \to \infty} \left| \frac{(-1)^{n+1} 2^{n+1} (x - 1)^{n+1}/(n+1)}{(-1)^n 2^n (x - 1)^n/n} \right| = \lim_{n \to \infty} \frac{2n|x - 1|}{n+1} \\
= \lim_{n \to \infty} \frac{2|x - 1|}{1 + 1/n} \\
= 2|x - 1|,
\]

so the series converges absolutely if \( 2|x - 1| < 1 \), or

\[
|x - 1| < \frac{1}{2}, \quad \frac{1}{2} < x < \frac{3}{2}.
\]

The power series diverges outside this interval of absolute convergence, where \( |x - 1| > 1/2 \).

- The radius of convergence is \( R = 1/2 \) and the center of the interval of convergence is \( a = 1 \).

- At the end point \( x = 3/2 \) where \( x - 1 = 1/2 \), the series becomes

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n}.
\]

This is an alternating harmonic series, which converges by the alternating series test. It does not converge absolutely since the harmonic series diverges, so the power series converges conditionally at \( x = 3/2 \).
• At the end point $x = 1/2$ where $x - 1 = -1/2$, the series becomes

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

This is a divergent harmonic series, so the power series diverges at $x = 1/2$. 
4. Write the Taylor polynomial $P_2(x)$ at $x = 0$ of order 2 for the function

$$f(x) = \ln(1 + x).$$

Use Taylor’s theorem with remainder to give a numerical estimate of the maximum error in approximating $\ln(1.1)$ by $P_2(0.1)$.

Solution.

- We have

$$f'(x) = \frac{1}{1 + x}, \quad f''(x) = -\frac{1}{(1 + x)^2}, \quad f'''(x) = \frac{2}{(1 + x)^3}.$$  

The first few Taylor coefficients of $f$ are therefore

$$c_0 = f(0) = 0, \quad c_1 = f'(0) = 1, \quad c_2 = \frac{f''(0)}{2!} = -\frac{1}{2}.$$  

Thus, the Taylor polynomial of $f$ at $x = 0$ of order 2 is

$$P_2(x) = c_0 + c_1 x + c_2 x^2 = x - \frac{1}{2} x^2.$$  

- By Taylor’s theorem with remainder,

$$\ln(1 + 0.1) = P_2(0.1) + R_2$$  

where

$$R_2 = \frac{f'''(c)(0.1)^3}{3!} = \frac{(0.1)^3}{3(1 + c)^3}$$  

for some $0 < c < 0.1$. Since $c > 0$, we have

$$0 < R_2 < \frac{(0.1)^3}{3}$$  

and therefore

$$P_2(0.1) < \ln(1.1) < P_2(0.1) + \frac{10^{-3}}{3}.$$  

- Remark. Since $P_2(0.1) = 0.095$ it follows that

$$0.095 < \ln(1.1) < 0.095334.$$  

The actual value to six decimal places is

$$\ln(1.1) = 0.095310.$$  

5. (a) Find the value(s) of $c$ for which the vectors
\[ \vec{u} = c\vec{i} + \vec{j} + c\vec{k}, \quad \vec{v} = 2\vec{i} - 3\vec{j} + c\vec{k}. \]
are orthogonal.
(b) Find the value(s) of $c$ for which the vectors
\[ \vec{u} = c\vec{i} + \vec{j} + c\vec{k}, \quad \vec{v} = 2\vec{i} - 3\vec{j} + c\vec{k}, \quad \vec{w} = \vec{i} + 6\vec{k}. \]
lie in the same plane.

Solution.

• (a) The vectors are orthogonal if their dot product is zero, or if
\[ \vec{u} \cdot \vec{v} = c \cdot 2 + 1 \cdot (-3) + c \cdot c = c^2 + 2c - 3 = (c + 3)(c - 1) = 0. \]
Hence, the vectors are orthogonal if
\[ c = -3 \text{ or } c = 1. \]

• (b) The vectors lie in the same plane if their scalar triple product is zero, or if
\[ \vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} c & 1 & c \\ 2 & -3 & c \\ 1 & 0 & 6 \end{vmatrix} \]
\[ = c \begin{vmatrix} -3 & c \\ 0 & 6 \end{vmatrix} - 2 \begin{vmatrix} 1 & c \\ 1 & 6 \end{vmatrix} + c \begin{vmatrix} 2 & -3 \\ 1 & 0 \end{vmatrix} \]
\[ = c(-18 - 0) - (12 - c) + c(0 + 3) \]
\[ = -14c - 12. \]
Hence, the vectors lie in the same plane if
\[ c = -\frac{6}{7}. \]
6. Find a parametric equation for the line in which the planes
\[ 3x - 6y - 4z = 15 \quad \text{and} \quad 6x + y - 2z = 5 \]
intersect.

Solution.

• Normal vectors to the planes are
\[ \vec{n}_1 = 3\vec{i} - 6\vec{j} - 4\vec{k}, \quad \vec{n}_2 = 6\vec{i} + \vec{j} - 2\vec{k}. \]
(The components of the normal vectors are the coefficients of the corresponding coordinates in the Cartesian equations of the planes.)

• A direction vector of the line of intersection is
\[ \vec{u} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -6 & -4 \\ 6 & 1 & -2 \end{vmatrix} \]
\[ = (12 + 4)\vec{i} - (-6 + 24)\vec{j} + (3 + 36)\vec{k} \]
\[ = 16\vec{i} - 18\vec{j} + 39\vec{k}. \]

• To find a point on the intersection of the planes, suppose that \( z = 0 \). Then
\[ 3x - 6y = 15, \quad 6x + y = 5 \]
whose solution is \( x = 15/13, \ y = -25/13 \).

• The parametric equation of the line is
\[ \vec{r} = \frac{15}{13}\vec{i} - \frac{25}{13}\vec{j} + t \left(16\vec{i} - 18\vec{j} + 39\vec{k}\right) \]
or
\[ x = \frac{15}{13} + 16t, \quad y = -\frac{25}{13} - 18t, \quad z = 39t. \]
7. Suppose that
\[ f(x, y) = e^x \cos \pi y \]
and
\[ x = u^2 - v^2, \quad y = u^2 + v^2. \]
*Using the chain rule*, compute the values of
\[ \frac{\partial f}{\partial u}, \quad \frac{\partial f}{\partial v} \]
at the point \((u, v) = (1, 1)\).

**Solution.**

- According to the chain rule
  \[ \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u}, \quad \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}. \]

- We have
  \[ \frac{\partial f}{\partial x} = e^x \cos \pi y, \quad \frac{\partial f}{\partial y} = -\pi e^x \sin \pi y \]
  and
  \[ \frac{\partial x}{\partial u} = 2u, \quad \frac{\partial x}{\partial v} = -2v, \quad \frac{\partial y}{\partial u} = 2u, \quad \frac{\partial y}{\partial v} = 2v. \]

- At \((u, v) = (1, 1)\) we have \((x, y) = (0, 2)\) and
  \[ \frac{\partial f}{\partial x} = e^0 \cos 2\pi = 1, \quad \frac{\partial f}{\partial y} = -\pi e^0 \sin 2\pi = 0, \]
  \[ \frac{\partial x}{\partial u} = 2, \quad \frac{\partial x}{\partial v} = -2, \quad \frac{\partial y}{\partial u} = 2, \quad \frac{\partial y}{\partial v} = 2. \]

- It follows that
  \[ \frac{\partial f}{\partial u} = 1 \cdot 2 + 0 \cdot 2 = 2, \quad \frac{\partial f}{\partial v} = 1 \cdot (-2) + 0 \cdot 2 = -2. \]
8. Let
\[ f(x, y, z) = \ln \left( x^2 + y^2 - 1 \right) + y + 6z. \]
In what direction \( \vec{u} \) is \( f(x, y, z) \) increasing most rapidly at the point \((1, 1, 0)\)? Give your answer as a unit vector \( \vec{u} \). What is the directional derivative of \( f \) in the direction \( \vec{u} \)?

**Solution.**

- We have
  \[
  \nabla f(x, y, z) = \left( \frac{2x}{x^2 + y^2 - 1} \right) \vec{i} + \left( \frac{2y}{x^2 + y^2 - 1} + 1 \right) \vec{j} + 6 \vec{k}.
  \]
  Thus
  \[
  \nabla f(1, 1, 0) = 2\vec{i} + 3\vec{j} + 6\vec{k}.
  \]
- The function \( f \) is most rapidly increasing in the direction
  \[
  \vec{u} = \frac{\nabla f}{|\nabla f|} = \frac{1}{7} \left( 2\vec{i} + 3\vec{j} + 6\vec{k} \right).
  \]
- The directional derivative of \( f \) in the direction \( \vec{u} \) is
  \[
  \left( \frac{df}{dt} \right)_{\vec{u}} = |\nabla f| = 7.
  \]
9. Find the equation of the tangent plane to the surface 
\[ xyz = 2 \]
at the point \((1, 1, 2)\).

**Solution.**

- The normal vector to the surface \( f(x, y, z) = 2 \) with \( f(x, y, z) = xyz \) is
  \( \nabla f(x, y, z) = yz\vec{i} + xz\vec{j} + xy\vec{k} \).
  
  Thus the normal vector at \((x, y, z) = (1, 1, 2)\) is
  \( \vec{n} = \nabla f(1, 1, 2) = 2\vec{i} + 2\vec{j} + \vec{k} \).

- The equation of the tangent plane to the surface at the point \((1, 1, 2)\) with position vector \( \vec{r}_0 = \vec{i} + \vec{j} + 2\vec{k} \) and normal \( \vec{n} \) is
  \[ \vec{n} \cdot (\vec{r} - \vec{r}_0) = 0. \]

  That is,
  \[ 2(x - 1) + 2(y - 1) + (z - 2) = 0 \]
or
  \[ 2x + 2y + z = 6. \]
10. Find all critical points of the function

\[ f(x, y) = x^4 - 8x^2 + 3y^2 - 6y. \]

and classify them as maximums, minimums, or saddle-point.

**Solution.**

- At a critical point
  \[ f_x = 4x^3 - 16x = 0, \quad f_y = 6y - 6 = 0. \]

  If follows that \( y = 1 \) and \( x = 0, \pm 2 \) so the critical points of \( f \) are
  \[(x, y) = (0, 1), (2, 1), (-2, 1). \]

- We have
  \[ f_{xx} = 12x^2 - 16, \quad f_{yy} = 6, \quad f_{xy} = 0. \]

  Hence
  \[ f_{xx}f_{yy} - f_{xy}^2 = (-16)(6) < 0 \quad \text{at} \ (x, y) = (0, 1) \]

  so \( f(x, y) \) has a saddle point at \((0, 1)\), and
  \[ f_{xx}f_{yy} - f_{xy}^2 = (32)(6) > 0, \quad \text{at} \ (x, y) = (\pm 2, 1). \]

  Since \( f_{yy} > 0 \), \( f(x, y) \) has minima at \((\pm 2, 1)\).
11. Let

\[ D = \{(x, y) : x^2 + y^2 \leq 1\} \]

be the unit disc and

\[ f(x, y) = x^2 - 2x + y^2 + 2y + 1. \]

Find the global maximum and minimum of

\[ f : D \to \mathbb{R} \]

At what points \((x, y)\) in \(D\) does \(f\) attain its maximum and minimum?

**Solution.**

- The function is differentiable everywhere. A critical point \((x, y)\) satisfies
  
  \[ f_x = 2x - 2 = 0, \quad 2y + 2 = 0 \]
  
  which implies that \((x, y) = (1, -1)\). This point does not lie inside \(D\), so
  \(f\) has no critical points inside \(D\) and it must attain its global maximum
  and minimum on the boundary of \(D\).

- On the boundary \(x^2 + y^2 = 1\), we can write
  
  \[ x = \cos \theta, \quad y = \sin \theta \]
  
  where \(0 \leq \theta \leq 2\pi\) is the polar angle of \((x, y)\). Then
  
  \[ f(\cos \theta, \sin \theta) = \cos^2 \theta - 2 \cos \theta + \sin^2 \theta + 2 \sin \theta + 1 = 2 - 2 \cos \theta + 2 \sin \theta. \]

- Since \(f(\cos \theta, \sin \theta)\) is a differentiable function of \(\theta\), the maximum and
  minimum of \(f\) on the boundary are attained at a critical point where
  
  \[ \frac{d}{d\theta} f(\cos \theta, \sin \theta) = 2 \sin \theta + 2 \cos \theta = 0. \]

  It follows that
  
  \[ \tan \theta = -1, \]

  so \(\theta = 3\pi/4\) or \(\theta = 7\pi/4\).
The second derivative of $f$ along the boundary is

$$\frac{d^2}{d^2 \theta} f(\cos \theta, \sin \theta) = 2 \cos \theta - 2 \sin \theta$$

We have

$$\cos \frac{3\pi}{4} = -\frac{1}{\sqrt{2}}, \quad \sin \frac{3\pi}{4} = \frac{1}{\sqrt{2}}, \quad \cos \frac{7\pi}{4} = \frac{1}{\sqrt{2}}, \quad \sin \frac{7\pi}{4} = -\frac{1}{\sqrt{2}}$$

so

$$\frac{d^2 f}{d^2 \theta} < 0 \quad \text{at} \quad \theta = 3\pi/4, \quad \frac{d^2 f}{d^2 \theta} > 0 \quad \text{at} \quad \theta = 7\pi/4.$$ Thus, $f$ has a maximum value at $\theta = 3\pi/4$ and a minimum value at $\theta = 7\pi/4$.

Evaluating the corresponding values of $(x, y)$ and $f$, we find that $f$ has a global maximum on $D$ at

$$(x, y) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \quad \text{where} \quad f(x, y) = 2 + 2\sqrt{2},$$

and a global minimum on $D$ at

$$(x, y) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \quad \text{where} \quad f(x, y) = 2 - 2\sqrt{2}.$$
12. Suppose that the material for the top and bottom of a rectangular box costs $a$ dollars per square meter and the material for the four sides costs $b$ dollars per square meter. Use the method of Lagrange multipliers to find the dimensions of a box of volume $V$ cubic meters that minimizes the cost of the materials used to construct it. What is the minimal cost?

Solution.

- Let $x$, $y$ be the lengths of the horizontal sides of the box and $z$ the height of the box.
- The cost of the material for the box is
  \[ C(x, y, z) = 2axy + 2bxz + 2byz. \]
  The volume of the box is $xyz$.
- Thus, we want to minimize $C(x, y, z)$ subject to the constraint that
  \[ g(x, y, z) = 0 \]
  where
  \[ g(x, y, z) = xyz - V. \]
- We have
  \[ \nabla C = (2ay + 2bz)\hat{i} + (2ax + 2bz)\hat{j} + (2bx + 2by)\hat{k}, \]
  \[ \nabla g = yz\hat{i} + xz\hat{j} + xy\hat{k} \]
- By the method of Lagrange multipliers, the equations for a critical point are $\nabla C = \lambda \nabla g$ and $g = 0$, or
  \[ 2ay + 2bz = \lambda yz, \quad 2ax + 2bz = \lambda xz, \quad 2bx + 2by = \lambda xy, \quad xyz = V. \]
- It follows from the first two equations that $x = y$, as one would expect by symmetry, and from the last equation that
  \[ z = \frac{V}{x^2}. \]
Hence, assuming that $x > 0$, we find that

\[ 2ax + \frac{2bV}{x^2} = \lambda V, \quad 4b = \lambda x \]

and, eliminating $\lambda$, we get

\[ 2ax = \frac{2bV}{x^2}. \]

It follows that

\[ x = y = \left( \frac{b}{a} \right)^{1/3} V^{1/3}, \quad z = \left( \frac{a}{b} \right)^{2/3} V^{1/3}. \]

**Remark.** Note that $a/b$ is a dimensionless ratio of costs per unit area and $V^{1/3}$ is a length, so these results have the correct dimensions (meters). If $a = b$, then $x = y = z$, which corresponds to the fact that the cube is the rectangular solid with minimal surface area that encloses a given volume. If $a > b$ then the box is taller than it is wide to reduce the amount of the more expensive material required for the top and bottom, while if $a < b$ then the box is shorter than it is wide to reduce the amount of material required for the sides.