CALCULUS: **Math 21C**
Spring 2018: Final Exam
Solutions

1. [10 pts] Do the following series converge or diverge? State clearly which test you use.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)}}$

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{1}{n}\right)$

Solution.

- (a) We have
$$0 \le \frac{1}{\sqrt{n(n+1)(n+2)}} \le \frac{1}{n^{3/2}},$$
so the series converges by comparison with the convergent p-series with $p = 3/2 > 1$.

- (b) Since $\cos x$ is continuous at $x = 0$, we have
$$\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos\left(\lim_{n\to\infty} \frac{1}{n}\right) = \cos 0 = 1,$$
so the terms in the series do not approach 0 as $n \to \infty$, and the series diverges by the nth-term test.

2. [20 pts] Determine the interval of convergence (including the endpoints) for the following power series. State explicitly for what values of x the series converges absolutely, converges conditionally, or diverges. Specify the radius of convergence R and the center of the interval of convergence a.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}(x+2)^n.$$

Solution.

- Let
$$a_n = \frac{(-1)^n}{n+1}(x+2)^n.$$
Then
$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{|x+2|^{n+1}}{n+2} \cdot \frac{n+1}{|x+2|^n} \\ &= |x+2|\frac{1+1/n}{1+2/n} \\ &\to |x+2| \qquad \text{as } n \to \infty.\end{aligned}$$

- By the ratio test, the series converges absolutely if $|x+2| < 1$ and diverges if $|x+2| > 1$. The radius of convergence is $R = 1$ and the center of the interval of convergence is $a = -2$.

- At the endpoint $x = -1$, the series becomes the alternating harmonic series
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1},$$
which converges conditionally. At the endpoint $x = -3$, the series becomes the harmonic series
$$\sum_{n=0}^{\infty} \frac{1}{n+1},$$
which diverges.

- The interval of convergence is $-3 < x \leq -1$.

3. [25 pts] (a) Write down the Taylor series centered at $x = 0$ for the functions e^x and $\sin x$, up to and including terms of degree 3 in x. (You don't need to derive the series.)

(b) Use the results in (a) to find the Taylor series centered at $x = 0$ for the function

$$f(x) = e^x \sin(2x),$$

up to and including terms of degree 3 in x.

(c) Use the result in (b) to evaluate $f'''(0)$.

Solution.

- (a) The Taylor series are

$$\begin{aligned} e^x &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots, \\ \sin x &= x - \frac{1}{6}x^3 + \dots. \end{aligned}$$

- (b) We have

$$\begin{aligned} e^x \sin(2x) &= \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots\right)\left(2x - \frac{1}{6}(2x)^3 + \dots\right) \\ &= 2x + 2x^2 + x^3 - \frac{8}{6}x^3 + \dots \\ &= 2x + 2x^2 - \frac{1}{3}x^3 + \dots \end{aligned}$$

- Using the expression for Taylor coefficients, we get that

$$\frac{f'''(0)}{3!} = -\frac{1}{3},$$

  so $f'''(0) = -2$.

4. [25 pts] Suppose that $P_0 = (1, 2, 3)$ and

$$f(x, y, z) = \frac{yz}{x}.$$

(a) Find an equation for the tangent plane to the surface $f(x, y, z) = 6$ at the point P_0.
(b) If $\vec{u} = \vec{i} + c\vec{j} - c\vec{k}$, where c is a constant, find the value of c such that the directional derivative $(D_{\vec{u}} f)_{P_0}$ at the point P_0 is equal to zero.

Solution.

- (a) We have
$$\nabla f(x, y, z) = -\frac{yz}{x^2}\vec{i} + \frac{z}{x}\vec{j} + \frac{y}{z}\vec{k}.$$
A normal vector for the tangent plane to the level surface of f at P_0 is given by $\vec{n} = \nabla f(1, 2, 3)$, or
$$\vec{n} = -6\vec{i} + 3\vec{j} + 2\vec{k}$$

- An equation for the plane is $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$, or
$$-6(x - 1) + 3(y - 2) + 2(z - 3) = 0$$
which gives $-6x + 3y + 2z = 6$.

- (b) The directional derivative is given by
$$\begin{aligned}(D_{\vec{u}} f)_{P_0} &= \nabla f(1, 2, 3) \cdot \vec{u} \\ &= \left(-6\vec{i} + 3\vec{j} + 2\vec{k}\right) \cdot \left(\vec{i} + c\vec{j} - c\vec{k}\right) \\ &= -6 + c,\end{aligned}$$
so the directional derivative is zero when $c = 6$.

5. [25 pts] Let C_1, C_2 be curves with parametric equations

$$C_1 : \vec{r}(t) = t^3\vec{i} + t\vec{j} + (t^2 - 1)\vec{k}, \qquad C_2 : \vec{r}(s) = (\cos s)\vec{i} + e^s\vec{j} + s\vec{k}.$$

(a) Show that the curves intersect at the point $P_0 = (1, 1, 0)$. For each curve, find a tangent vector at P_0.

(b) Find a parametric equation for the line through P_0 that is orthogonal to both curves at P_0.

Solution.

- (a) For C_1, we have $\vec{r}(t) = \vec{i}+\vec{j}$ at $t = 1$, and for C_2, we have $\vec{r}(s) = \vec{i}+\vec{j}$ at $s = 0$, so the curves intersect at $(1, 1, 0)$.

- A tangent vector to C_1 is $\vec{r}'(t) = 3t^2\vec{i} + \vec{j} + 2t\vec{k}$, so a tangent vector $\vec{t}_1 = \vec{r}'(1)$ at P_0 is

$$\vec{t}_1 = 3\vec{i} + \vec{j} + 2\vec{k}$$

- A tangent vector to C_2 is $\vec{r}'(s) = (-\sin s)\vec{i}+e^s\vec{j}+\vec{k}$, so a tangent vector $\vec{t}_2 = \vec{r}'(0)$ at P_0 is

$$\vec{t}_2 = \vec{j} + \vec{k}.$$

- (b) A direction vector $\vec{v}$ of the line orthogonal to $\vec{t}_1$ and $\vec{t}_2$ is given by

$$\vec{v} = \vec{t}_1 \times \vec{t}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = -\vec{i} - 3\vec{j} + 3\vec{k}.$$

- A parametric equation $\vec{r}(t) = \vec{r}_0 + t\vec{v}$ of the line is

$$x = 1 - t, \qquad y = 1 - 3t, \qquad z = 3t.$$

6. [20 pts] (a) Suppose that $w(s,t) = f(x(s,t), y(s,t))$. Write down the chain rule for the partial derivatives w_s, w_t.
(b) Use the chain rule to compute the partial derivatives w_s, w_t of $w(s,t) = f(x(s,t), y(s,t))$ if

$$f(x,y) = xe^y + \sin(xy), \qquad x(s,t) = s^2 t, \qquad y(s,t) = s - t.$$

You should write your answers as functions of (s,t), but you don't need to simplify them.

Solution.

- (a) The chain rule is

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}, \qquad \frac{\partial w}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

- (b) We have

$$\begin{aligned}
\frac{\partial w}{\partial s} &= [e^y + y\cos(xy)] \cdot 2st + [xe^y + x\cos(xy)] \cdot 1 \\
&= \left\{e^{s-t} + (s-t)\cos[s^2t(s-t)]\right\} \cdot 2st + s^2te^{s-t} + s^2t\cos\left[s^2t(s-t)\right], \\
\frac{\partial w}{\partial t} &= [e^y + y\cos(xy)] \cdot s^2 + [xe^y + x\cos(xy)] \cdot (-1) \\
&= \left\{e^{s-t} + (s-t)\cos[s^2t(s-t)]\right\} \cdot s^2 - \left\{s^2te^{s-t} + s^2t\cos\left[s^2t(s-t)\right]\right\}.
\end{aligned}$$

7. [25 pts] (a) Find all critical points of the function

$$f(x, y) = x^2y - x^2 - 2y^2.$$

(b) Classify the critical points as local maximums, local minimums, or saddle-points.

Solution.

- (a) We have $f_x = 2xy - 2x$, $f_y = x^2 - 4y$, so the critical points are solutions of
$$2xy - 2x = 0, \qquad x^2 - 4y = 0.$$
From the first equation, either $x = 0$ or $y = 1$, and then from the second equation $y = 0$ or $x = \pm 2$, so the critical point are
$$(0, 0), \qquad (-2, 1), \qquad (2, 1).$$

- (b) We have $f_{xx} = 2y - 2$, $f_{xy} = 2x$, $f_{yy} = -4$, and the discriminant is
$$D = \begin{vmatrix} 2y - 2 & 2x \\ 2x & -4 \end{vmatrix} = -8y + 8 - 4x^2.$$

- At $(0, 0)$, we have $f_{xx} = -2 < 0$ and $D = 8 > 0$, so f has a local maximum.

- At $(\pm 2, 1)$, we have $D = -16 < 0$, so f has saddle-points.

8. [20 pts] Use the method of Lagrange multipliers to find the maximum and minimum values of the function

$$f(x, y, z) = 2x + 3y + 6z$$

subject to the constraint

$$x^2 + y^2 + z^2 = 1.$$

Solution.

- The constraint function is

$$g(x, y, z) = x^2 + y^2 + z^2 - 1.$$

- The Lagrange multiplier equation $\nabla f = \lambda \nabla g$ gives

$$2 = 2\lambda x, \qquad 3 = 2\lambda y, \qquad 6 = 2\lambda z,$$

  so

$$x = \frac{2}{2\lambda}, \qquad y = \frac{3}{2\lambda}, \qquad z = \frac{6}{2\lambda}.$$

- Using these equations in the constraint equation $g(x, y, z) = 0$, we get

$$\frac{2^2 + 3^2 + 6^2}{(2\lambda)^2} = 1,$$

  which implies that $(2\lambda)^2 = 49$ or $2\lambda = \pm 7$. It follows that the two critical points are

$$(x, y, z) = \left(\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right), \qquad (x, y, z) = \left(-\frac{2}{7}, -\frac{3}{7}, -\frac{6}{7}\right).$$

- We have

$$f\left(\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right) = 7, \qquad f\left(-\frac{2}{7}, -\frac{3}{7}, -\frac{6}{7}\right) = -7,$$

  so the maximum value of f is 7 and the minimum value is -7.

9. [10 pts] If $\vec{u}$, $\vec{v}$, $\vec{w}$ are any three vectors in space, show that

$$\vec{u} \times (\vec{v} \times \vec{w}) + \vec{v} \times (\vec{w} \times \vec{u}) + \vec{w} \times (\vec{u} \times \vec{v}) = 0.$$

HINT. You can use the following identity that we discussed in class:

$$\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}.$$

Solution.

- Cyclically permuting $\vec{u}$, $\vec{v}$, $\vec{w}$, we get that

$$\begin{aligned}
\vec{u} \times (\vec{v} \times \vec{w}) &= (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w} \\
\vec{v} \times (\vec{w} \times \vec{u}) &= (\vec{v} \cdot \vec{u})\vec{w} - (\vec{v} \cdot \vec{w})\vec{u} \\
\vec{w} \times (\vec{u} \times \vec{v}) &= (\vec{w} \cdot \vec{v})\vec{u} - (\vec{w} \cdot \vec{u})\vec{v}
\end{aligned}$$

- Since $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$, it follows that

$$\begin{aligned}
\vec{u} \times (\vec{v} \times \vec{w}) + \vec{v} \times (\vec{w} \times \vec{u}) + \vec{w} \times (\vec{u} \times \vec{v}) &= (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w} \\
&+ (\vec{v} \cdot \vec{u})\vec{w} - (\vec{v} \cdot \vec{w})\vec{u} \\
&+ (\vec{w} \cdot \vec{v})\vec{u} - (\vec{w} \cdot \vec{u})\vec{v} \\
&= (\vec{u} \cdot \vec{w})\vec{v} - (\vec{w} \cdot \vec{u})\vec{v} \\
&+ (\vec{v} \cdot \vec{u})\vec{w} - (\vec{u} \cdot \vec{v})\vec{w} \\
&+ (\vec{w} \cdot \vec{v})\vec{u} - (\vec{v} \cdot \vec{w})\vec{u} \\
&= 0.
\end{aligned}$$