Calculus: Midterm 1
Math 21C, Spring 2018
Solutions

1. [20pts] Do the following sequences $\{a_n\}$ converge or diverge as $n \to \infty$? If a sequence converges, find its limit. Justify your answers.

(a) $a_n = \dfrac{3n}{\sqrt{n^2+2}}$

(b) $a_n = \cos\left(\dfrac{1}{\sqrt{n}}\right)$

(c) $a_n = 1 + (-1)^n$

(d) $a_n = \dfrac{n!}{n^n}$

Solution.

- (a) We have
$$\lim_{n\to\infty} \frac{3n}{\sqrt{n^2+2}} = \lim_{n\to\infty} \frac{3}{\sqrt{1+2/n^2}} = 3.$$

- (b) Since $\cos x$ is a continuous function, we have
$$\lim_{n\to\infty} \cos\left(\frac{1}{\sqrt{n}}\right) = \cos\left(\lim_{n\to\infty} \frac{1}{\sqrt{n}}\right) = \cos 0 = 1.$$

- (c) The sequence diverges since its terms oscillate between 0 (for n odd) and 2 (for n even).

- (d) We have
$$\begin{aligned} a_n &= \frac{1\cdot 2\cdot 3\ldots n}{n\cdot n\cdot n\ldots n} \\ &= \frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}\cdots\frac{n}{n} \\ &\le \frac{1}{n}. \end{aligned}$$

  Since $0 \le a_n \le 1/n$ and $1/n \to 0$, we have $a_n \to 0$ by the sandwich theorem.

2. [10pts] Find the sums of the following series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^{n+1}}{3^n}$

(b) $\displaystyle\sum_{n=2}^{\infty} \left(\frac{n-1}{n} - \frac{n}{n+1}\right)$

Solution.

- (a) This is a geometric series with ratio $r = -2/3$. Since $|r| < 1$, the series converges and the sum is

$$\sum_{n=1}^{\infty} \frac{(-2)^{n+1}}{3^n} = \frac{4}{3} \sum_{n=0}^{\infty} \left(\frac{-2}{3}\right)^n = \frac{4}{3}\left[\frac{1}{1-(-2/3)}\right] = \frac{4}{5}$$

- (b) This is a telescoping series of the form

$$\sum_{n=2}^{\infty} (b_n - b_{n+1}), \qquad b_n = \frac{n-1}{n}.$$

The partial sums are given by

$$\sum_{k=2}^{n} (b_k - b_{k+1}) = b_2 - b_{n+1} = \frac{1}{2} - \frac{n}{n+1}.$$

We have $n/(n+1) \to 1$ as $n \to \infty$, so

$$\sum_{n=2}^{\infty} \left(\frac{n-1}{n} - \frac{n}{n+1}\right) = \frac{1}{2} - 1 = -\frac{1}{2}.$$

3. [30pts] Determine whether the following series converge absolutely, converge conditionally, or diverge. You can use any appropriate test provided that you explain your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{8^n}{7^n + 9}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{(2n + 7)^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n + 3}{3n + 1}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}}$

(e) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n^5 + 2}}$

(f) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n^2 + 5}}$

Solution.

- (a) Let
$$a_n = \frac{8^n}{7^n + 9}.$$
Then
$$\frac{a_{n+1}}{a_n} = \frac{8}{7}\left(\frac{1 + 9/7^n}{1 + 9/7^{n+1}}\right) \to \frac{8}{7} > 1 \qquad \text{as } n \to \infty,$$
so the series $\sum a_n$ diverges by the ratio test.

- (b) Let
$$a_n = \frac{5}{(2n + 7)^n}, \qquad \sqrt[n]{a_n} = \frac{\sqrt[n]{5}}{2n + 7} \to 0 \qquad \text{as } n \to \infty.$$
Since $\lim \sqrt[n]{a_n} < 1$, the series converges by the root test.

- (c) The limit of the terms is

$$\lim_{n\to\infty} \frac{n+3}{3n+1} = \lim_{n\to\infty} \frac{1+3/n}{3+1/n} = \frac{1}{3}.$$

  Since this limit is nonzero, the series diverges by the nth-term test.

- (d) We have $\ln n \geq 1$ for $n \geq 3$, so

$$\frac{\ln n}{\sqrt{n}} \geq \frac{1}{\sqrt{n}} \qquad \text{for } n \geq 3.$$

  Since the p-series with $p = 1/2 < 1$ diverges, the comparison test implies that that the series $\sum \ln n/\sqrt{n}$ diverges.

- (e) Let

$$a_n = (-1)^{n+1} \frac{1}{\sqrt{n^5+2}}, \qquad b_n = \frac{1}{n^{5/2}}.$$

  Then

$$\lim_{n\to\infty} \left| \frac{a_n}{b_n} \right| = \lim_{n\to\infty} \frac{n^{5/2}}{\sqrt{n^5+2}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+2/n^5}} = 1.$$

  Since $\sum b_n$ is a convergent p-series with $p = 5/2 > 1$, the series $\sum a_n$ converges absolutely by the limit comparison test.

- (f) The series is of the form $\sum (-1)^{n+1} u_n$ where

$$u_n = \frac{1}{\sqrt{n^2+5}}$$

  is a positive, decreasing sequence with limit zero, so the alternating series test implies that the series converges. However, since

$$\lim_{n\to\infty} \frac{\sum 1/\sqrt{n^2+5}}{1/n} = 1,$$

  the limit comparison test with the divergent harmonic series $\sum 1/n$ shows that the series $\sum 1/\sqrt{n^2+5}$ diverges, so the original series converges conditionally.

4. [10pts] Use the integral test to show that the series

$$\sum_{n=1}^{\infty} ne^{-n^2}$$

converges, and show that its sum is less than 1/2.

Solution.

- Let $f(x) = xe^{-x^2}$, so the nth term in the series is $f(n)$. Then $f(x)$ is a continuous, positive function for $x \geq 1$, and

$$f'(x) = \left(1 - 2x^2\right) e^{-x^2} < 0 \qquad \text{for } x > 1/\sqrt{2},$$

so $f(x)$ is a decreasing for $x \geq 1$. Making the substitution $u = x^2$, we get that

$$\int_1^{\infty} f(x)\, dx = \lim_{b\to\infty} \int_1^b xe^{-x^2}\, dx = \frac{1}{2} \lim_{b\to\infty} \int_1^{b^2} e^{-u}\, du = \frac{1}{2e},$$

so the integral converges and the integral test implies that the series converges.

- By considering lower Riemann sums, we see that

$$\sum_{n=3}^{\infty} ne^{-n^2} \leq \int_2^{\infty} f(x)\, dx = \frac{1}{2} \int_4^{\infty} e^{-u}\, du = \frac{1}{2e^4}.$$

It follows that

$$\sum_{n=1}^{\infty} ne^{-n^2} = \frac{1}{e} + \frac{2}{e^4} + \sum_{n=3}^{\infty} ne^{-n^2} \leq \frac{1}{e} + \frac{5}{2e^4}.$$

Since $e = \sum_{n=0}^{\infty} 1/n! > 1 + 1 + 1/2! = 5/2$, we have

$$\frac{1}{e} + \frac{5}{2e^4} < \frac{2}{5} + \left(\frac{2}{5}\right)^3 = \frac{58}{125} < \frac{1}{2},$$

so the sum of the series is less than 1/2.

Remark. The last part of this problem was more complicated than intended. It would be much easier to estimate the sum of the series starting from $n = 2$:

$$\sum_{n=2}^{\infty} ne^{-n^2} < \int_1^{\infty} xe^{-x^2}\, dx = \frac{1}{2e}.$$