Solutions: Midterm 2
Math 21C, Spring 2018

1. [20%] Determine the interval of convergence (including the endpoints) for the following power series. State explicitly for what values of x the series converges absolutely, converges conditionally, or diverges. Give the radius of convergence R and the center of the interval of convergence a.

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{n^2+1}$$

Solution.

- Let
$$a_n = \frac{(-3)^n x^n}{n^2+1}.$$
- Then
$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \frac{3^{n+1}|x|^{n+1}}{(n+1)^2+1} \cdot \frac{n^2+1}{3^n|x|^n} \\ &= 3|x| \cdot \frac{n^2+1}{(n+1)^2+1} \\ &= 3|x| \cdot \frac{1+1/n^2}{(1+1/n)^2+1/n^2} \\ &\to 3|x| \qquad \text{as } n \to \infty. \end{aligned}$$
- The ratio test implies that the series $\sum a_n$ converges absolutely if $3|x| < 1$ and diverges if $3|x| > 1$. The center of the interval of convergence is $a = 0$ and the radius of convergence is $R = 1/3$.
- At $x = \pm 1/3$, we have
$$|a_n| = \frac{1}{n^2+1} \leq \frac{1}{n^2} \qquad \text{for } n \geq 1,$$
so the series converges absolutely at $x = \pm 1/3$ by comparison with the convergent p-series with $p = 2$. The interval of convergence is $-1/3 \leq x \leq 1/3$.

2. [20%] Let

$$\vec{u} = 2\vec{i} - \vec{j} + \vec{k}, \qquad \vec{v} = 4\vec{i} + 3\vec{k}.$$

(a) Compute $\vec{u} \cdot \vec{v}$.

(b) Compute $\vec{u} \times \vec{v}$.

(c) Find the angle θ between $\vec{u}$ and $\vec{v}$.

Solution.

- (a) We have

$$\vec{u} \cdot \vec{v} = 2 \cdot 4 - 1 \cdot 0 + 1 \cdot 3 = 11.$$

- (b) We have

$$\begin{aligned} \vec{u} \times \vec{v} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 4 & 0 & 3 \end{vmatrix} \\ &= \begin{vmatrix} -1 & 1 \\ 0 & 3 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & -1 \\ 4 & 0 \end{vmatrix} \vec{k} \\ &= -3\vec{i} - 2\vec{j} + 4\vec{k} \end{aligned}$$

(c) We have

$$\begin{aligned} |\vec{u}|^2 &= 2^2 + (-1)^2 + 1^2 = 6, \\ |\vec{v}|^2 &= 4^2 + 0^2 + 3^2 = 25, \end{aligned}$$

so $|\vec{u}| = \sqrt{6}$, $|\vec{v}| = 5$, and

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|} = \frac{11}{5\sqrt{6}}, \qquad \theta = \cos^{-1}\left(\frac{11}{5\sqrt{6}}\right).$$

3. [20%] Suppose that L_1, L_2 are lines with the parametric equations

$$\begin{aligned} L_1: &\qquad x = 4 + 2t,\ \ y = 1 + t,\ \ z = 2 + t; \\ L_2: &\qquad x = 4 + t,\ \ \ y = 1 - t,\ \ z = 2 + 3t. \end{aligned}$$

Find an equation for the plane that contains L_1 and L_2.

Solution.

- The lines intersect at the point $P_0(4, 1, 2)$, so they are contained in the plane through P_0 whose normal vector $\vec{n}$ is orthogonal to the direction vectors of both lines.

- The direction vectors of L_1, L_2 are

  $$\vec{v}_1 = 2\vec{i} + \vec{j} + \vec{k}, \qquad \vec{v}_2 = \vec{i} - \vec{j} + 3\vec{k},$$

  so a normal vector to the plane is

  $$\begin{aligned} \vec{n} &= \vec{v}_1 \times \vec{v}_2 \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 1 & -1 & 3 \end{vmatrix} \\ &= 4\vec{i} - 5\vec{j} - 3\vec{k} \end{aligned}$$

- The equation of the plane is $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$, or

  $$4(x - 4) - 5(y - 1) - 3(z - 2) = 0,$$

  which gives

  $$4x - 5y - 3z = 5.$$

4. [20%] Let

$$f(x, y) = \sin(x + 2y) + x^3 y.$$

Compute the partial derivatives f_x, f_y, f_{xx}, f_{xy}, f_{yy}.

Solution.

- We have

$$\begin{aligned} f_x &= \cos(x + 2y) + 3x^2 y \\ f_y &= 2\cos(x + 2y) + x^3 \\ f_{xx} &= -\sin(x + 2y) + 6xy \\ f_{xy} &= -2\sin(x + 2y) + 3x^2 \\ f_{yy} &= -4\sin(x + 2y) \end{aligned}$$

5. [20%] (a) Find the Taylor polynomial $P_2(x)$ of order 2 at $x = 0$ for the function $f(x) = xe^{-x}$.
(b) Write down an expression for the remainder $R_2(x)$ and use it to find an estimate for the maximum error between $f(x)$ and $P_2(x)$ when $x = 0.1$.

Solution.

- (a) We have

  $$f'(x) = (1-x)e^{-x}, \qquad f''(x) = (x-2)e^{-x}, \qquad f'''(x) = (3-x)e^{-x}.$$

  The first three Taylor coefficients of $f(x)$ at $x = 0$ are

  $$c_0 = f(0) = 0, \qquad c_1 = f'(0) = 1, \qquad c_2 = \frac{f''(0)}{2!} = -1,$$

  so

  $$P_2(x) = x - x^2.$$

- Alternatively, since $e^{-x} = 1 - x + \ldots$, we have $xe^{-x} = x - x^2 + \ldots$.

- (b) The remainder is given by

  $$R_2(x) = \frac{f'''(c)}{3!}x^3, \qquad f'''(c) = (3-c)e^{-c}.$$

  for some $0 < c < x$. Both factors $(3-c)$ and e^{-c} are positive and decreasing for $0 < c < 3$, so $f'''(c)$ attains its maximum value on $0 \le c \le 3$ at the left endpoint $c = 0$, meaning that $0 \le f'''(c) \le 3$.

- It follows that

  $$0 \le f(0.1) - P_2(0.1) \le \frac{1}{2}(0.1)^3.$$