Calculus: **Math 21C, Spring 2018**
Sample Final Questions: Solutions

1. Do the following sequences $\{a_n\}$ converge or diverge as $n \to \infty$? If a sequence converges, find its limit. Justify your answers.

$$\text{(a)} \quad a_n = \frac{2n^2 + 3n^3}{2n^3 + 3n^2}; \qquad \text{(b)} \quad a_n = \frac{\sin(n^2)}{\sqrt{n}}; \qquad \text{(c)} \quad a_n = \frac{n}{\ln n}.$$

Solution.

- (a) Dividing the numerator and denominator by n^3, we have

$$a_n = \frac{2/n + 3}{2 + 3/n} \to \frac{3}{2} \qquad \text{as } n \to \infty$$

  so the sequence converges to 3/2.

- (b) Since $|\sin x| \le 1$, we have

$$|a_n| \le \frac{1}{\sqrt{n}},$$

  and $1/\sqrt{n} \to 0$ as $n \to \infty$, so the sequence converges to 0 by the sandwich theorem.

- (c) Since both numerator and denominator diverge to infinity, we can apply l'Hôspital's rule to get that

$$\lim_{n\to\infty} \frac{n}{\ln n} = \lim_{x\to\infty} \frac{x}{\ln x} = \lim_{x\to\infty} \frac{1}{1/x} = \lim_{x\to\infty} x = \infty,$$

  so the sequence diverges to infinity.

2. Do the following series converge or diverge? State clearly which test you use.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n+4}{6n-17}$

(b) $\displaystyle\sum_{n=1}^{\infty} \sqrt{\frac{n}{n^4+7}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^{n+1}}{(2n)!}$

(d) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n}$

(e) $\displaystyle\frac{1}{1^4} + \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} - \frac{1}{6^4} + \frac{1}{7^4} - \frac{1}{9^4} + \cdots$

(f) $\displaystyle\sum_{n=1}^{\infty} \left[e^n - e^{n+1}\right]$

Solution.

- We write each series as $\sum c_n$.
- (a) We have
$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{n+4}{6n-17} = \frac{1}{6}.$$
Since this limit is nonzero, the series diverges by the nth term test.
- (b) We have
$$0 \le c_n = \sqrt{\frac{n}{n^4+7}} \le \sqrt{\frac{n}{n^4}} = \frac{1}{n^{3/2}}.$$
Therefore the series converges by the comparison test, since the p-series with $p = 3/2 > 1$ converges.

- (c) We have

$$\begin{aligned}\left|\frac{c_{n+1}}{c_n}\right| &= \left|\frac{(-5)^{n+2}/(2n+2)!}{(-5)^{n+1}/(2n)!}\right| \\ &= \frac{5(2n)!}{(2n+2)!} \\ &= \frac{5}{(2n+1)(2n+2)}.\end{aligned}$$

  It follows that

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = 0.$$

  Since this limit exists and is less than 1, the series converges absolutely by the ratio test, and therefore it converges.

- (d) For $x \geq 3$, we have $\ln x > 1$, and

$$\frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{1-\ln x}{x^2} < 0,$$

  so $\ln n/n$ is decreasing for $n \geq 3$, and $\ln n/n \to 0$ as $n \to \infty$ by l'Hôspital's rule. Hence, the series converges by the alternating series test.

- (e) We have

$$|c_n| = \frac{1}{n^4}$$

  so the series converges absolutely, since the p-series with $p = 4$ converges. Therefore the series converges since any absolutely convergent series is convergent.

- (f) Either note that

$$c_n = e^n - e^{n+1} = e^n\,(1-e)$$

  diverges to $-\infty$ as $n \to \infty$, so the series diverges by the nth term test. Or note that the series is a telescoping series of the form $c_n = b_n - b_{n+1}$ with $b_n = e^n$ and the limit of b_n as $n \to \infty$ does not exist, so the series is a divergent telescoping series.

3. Determine the interval of convergence (including the endpoints) for the following power series. State explicitly for what values of x the series converges absolutely, converges conditionally, and diverges. Specify the radius of convergence R and the center of the interval of convergence a.

$$\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n} (x-1)^n.$$

Solution.

- Applying the ratio test, we have

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{(-1)^{n+1} 2^{n+1} (x-1)^{n+1}/(n+1)}{(-1)^n 2^n (x-1)^n/n} \right| &= \lim_{n\to\infty} \frac{2n|x-1|}{n+1} \\
&= \lim_{n\to\infty} \frac{2|x-1|}{1+1/n} \\
&= 2|x-1|,
\end{aligned}$$

  so the series converges absolutely if $2|x-1| < 1$, or

$$|x-1| < \frac{1}{2}, \qquad \frac{1}{2} < x < \frac{3}{2}.$$

  The power series diverges outside this interval of absolute convergence, where $|x-1| > 1/2$.

- The radius of convergence is $R = 1/2$ and the center of the interval of convergence is $a = 1$.

- At the end point $x = 3/2$ where $x - 1 = 1/2$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

  This is an alternating harmonic series, which converges by the alternating series test. It does not converge absolutely since the harmonic series diverges, so the power series converges conditionally at $x = 3/2$.

- At the end point $x = 1/2$ where $x - 1 = -1/2$, the series becomes

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

  This is a divergent harmonic series, so the power series diverges at $x = 1/2$.

- It follows that the interval of convergence is $1/2 < x \leq 3/2$.

4. Write the Taylor polynomial $P_2(x)$ at $x = 0$ of order 2 for the function

$$f(x) = \ln(1 + x).$$

Use Taylor's theorem with remainder to give a numerical estimate of the maximum error in approximating $\ln(1.1)$ by $P_2(0.1)$.

Solution.

- We have

  $$f'(x) = \frac{1}{1+x}, \qquad f''(x) = -\frac{1}{(1+x)^2}, \qquad f'''(x) = \frac{2}{(1+x)^3}.$$

  The first few Taylor coefficients of f are therefore

  $$c_0 = f(0) = 0, \qquad c_1 = f'(0) = 1, \qquad c_2 = \frac{f''(0)}{2!} = -\frac{1}{2}.$$

  Thus, the Taylor polynomial of f at $x = 0$ of order 2 is

  $$P_2(x) = c_0 + c_1 x + c_2 x^2 = x - \frac{1}{2}x^2.$$

- By Taylor's theorem with remainder,

  $$\ln(1 + 0.1) = P_2(0.1) + R_2$$

  where

  $$R_2 = \frac{f'''(c)}{3!}(0.1)^3 = \frac{(0.1)^3}{3(1+c)^3}$$

  for some $0 < c < 0.1$. Since $c > 0$, we have

  $$0 < R_2 < \frac{(0.1)^3}{3}$$

  and therefore

  $$P_2(0.1) < \ln(1.1) < P_2(0.1) + \frac{10^{-3}}{3}.$$

- **Remark.** Since $P_2(0.1) = 0.095$ it follows that

  $$0.095 < \ln(1.1) < 0.095334.$$

  The actual value to six decimal places is

  $$\ln(1.1) = 0.095310.$$

5. (a) Find the value(s) of c for which the vectors

$$\vec{u} = c\vec{i} + \vec{j} + c\vec{k}, \qquad \vec{v} = 2\vec{i} - 3\vec{j} + c\vec{k}.$$

are orthogonal.
(b) Find the value(s) of c for which the vectors

$$\vec{u} = c\vec{i} + \vec{j} + c\vec{k}, \qquad \vec{v} = 2\vec{i} - 3\vec{j} + c\vec{k}, \qquad \vec{w} = \vec{i} + 6\vec{k}.$$

lie in the same plane.

Solution.

- (a) The vectors are orthogonal if their dot product is zero, or if

  $$\vec{u} \cdot \vec{v} = c \cdot 2 + 1 \cdot (-3) + c \cdot c = c^2 + 2c - 3 = (c+3)(c-1) = 0.$$

  Hence, the vectors are orthogonal if

  $$c = -3 \text{ or } c = 1.$$

- (b) The vectors lie in the same plane if their scalar triple product is zero, or if

  $$\begin{aligned} \vec{u} \cdot (\vec{v} \times \vec{w}) &= \begin{vmatrix} c & 1 & c \\ 2 & -3 & c \\ 1 & 0 & 6 \end{vmatrix} \\ &= c \begin{vmatrix} -3 & c \\ 0 & 6 \end{vmatrix} - \begin{vmatrix} 2 & c \\ 1 & 6 \end{vmatrix} + c \begin{vmatrix} 2 & -3 \\ 1 & 0 \end{vmatrix} \\ &= c\,(-18 - 0) - (12 - c) + c\,(0 + 3) \\ &= -14c - 12. \end{aligned}$$

  Hence, the vectors lie in the same plane if

  $$c = -\frac{6}{7}.$$

6. Find an equation for the plane that is orthogonal to the curve

$$\vec{r}(t) = t^2\vec{i} + (2t - 1)\vec{j} + t^3\vec{k}$$

at the point $(1, 1, 1)$.

Solution.

- The tangent vector to the curve is

$$\vec{r}\,'(t) = 2t\vec{i} + 2\vec{j} + 3t^2\vec{k}.$$

- We have

$$\vec{r}(1) = (1, 1, 1), \qquad \vec{r}\,'(1) = 2\vec{i} + 2\vec{j} + 3\vec{k},$$

  and the required plane is the plane that passes through $\vec{r}(1)$ with normal vector $\vec{r}\,'(1)$.

- The equation of the plane is

$$\langle 2, 2, 3\rangle \cdot \langle x - 1, y - 1, z - 1\rangle = 0,$$

  or

$$2x + 2y + 3z = 7.$$

7. Suppose that

$$f(x,y) = e^x \cos \pi y$$

and

$$x = u^2 - v^2, \qquad y = u^2 + v^2.$$

Using the chain rule, compute the values of

$$\frac{\partial f}{\partial u}, \qquad \frac{\partial f}{\partial v}$$

at the point $(u,v) = (1,1)$.

Solution.

- According to the chain rule

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \qquad \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

- We have

$$\frac{\partial f}{\partial x} = e^x \cos \pi y, \qquad \frac{\partial f}{\partial y} = -\pi e^x \sin \pi y$$

and

$$\frac{\partial x}{\partial u} = 2u, \quad \frac{\partial x}{\partial v} = -2v, \qquad \frac{\partial y}{\partial u} = 2u, \quad \frac{\partial y}{\partial v} = 2v.$$

- At $(u,v) = (1,1)$ we have $(x,y) = (0,2)$ and

$$\frac{\partial f}{\partial x} = e^0 \cos 2\pi = 1, \qquad \frac{\partial f}{\partial y} = -\pi e^0 \sin 2\pi = 0,$$
$$\frac{\partial x}{\partial u} = 2, \quad \frac{\partial x}{\partial v} = -2, \qquad \frac{\partial y}{\partial u} = 2, \quad \frac{\partial y}{\partial v} = 2.$$

- It follows that

$$\frac{\partial f}{\partial u} = 1 \cdot 2 + 0 \cdot 2 = 2, \qquad \frac{\partial f}{\partial v} = 1 \cdot (-2) + 0 \cdot 2 = -2.$$

8. Let

$$f(x,y,z) = \ln\left(x^2+y^2-1\right) + y + 6z.$$

In what direction $\vec{u}$ is $f(x,y,z)$ increasing most rapidly at the point $(1,1,0)$? Give your answer as a unit vector $\vec{u}$. What is the directional derivative of f in the direction $\vec{u}$?

Solution.

- We have

  $$\nabla f(x,y,z) = \left(\frac{2x}{x^2+y^2-1}\right)\vec{i} + \left(\frac{2y}{x^2+y^2-1}+1\right)\vec{j} + 6\vec{k}.$$

  Thus

  $$\nabla f(1,1,0) = 2\vec{i} + 3\vec{j} + 6\vec{k}.$$

- The function f is most rapidly increasing in the direction

  $$\vec{u} = \frac{\nabla f}{|\nabla f|} = \frac{1}{7}\left(2\vec{i} + 3\vec{j} + 6\vec{k}\right).$$

- The directional derivative of f in the direction $\vec{u}$ is

  $$\left(\frac{df}{dt}\right)_{\vec{u}} = |\nabla f| = 7.$$

9. Find a parametric equation for the line that is orthogonal to the surface $xyz = 2$ at the point $(1, 1, 2)$.

Solution.

- The normal vector to the surface $f(x, y, z) = 2$ with $f(x, y, z) = xyz$ is

$$\nabla f(x, y, z) = yz\vec{i} + xz\vec{j} + xy\vec{k}.$$

  Thus, the normal vector at $(x, y, z) = (1, 1, 2)$ is

$$\vec{n} = \nabla f(1, 1, 2) = 2\vec{i} + 2\vec{j} + \vec{k}.$$

- The normal vector $\vec{n} = \langle 2, 2, 1\rangle$ is a direction vector of the line, so a parametric equation of the line is

$$\langle x, y, z\rangle = \langle 1, 1, 2\rangle + t\langle 2, 2, 1\rangle,$$

  or

$$x = 1 + 2t, \qquad y = 1 + 2t, \qquad z = 2 + t.$$

10. Find all critical points of the function

$$f(x, y) = x^4 - 8x^2 + 3y^2 - 6y.$$

and classify them as maximums, minimums, or saddle-point.

Solution.

- At a critical point

  $$f_x = 4x^3 - 16x = 0, \qquad f_y = 6y - 6 = 0.$$

  If follows that $y = 1$ and $x = 0, \pm 2$ so the critical points of f are

  $$(x, y) = (0, 1), (2, 1), (-2, 1).$$

- We have

  $$f_{xx} = 12x^2 - 16, \qquad f_{yy} = 6, \qquad f_{xy} = 0.$$

  Hence

  $$f_{xx}f_{yy} - f_{xy}^2 = (-16)(6) < 0 \qquad \text{at } (x, y) = (0, 1)$$

  so $f(x, y)$ has a saddle point at $(0, 1)$, and

  $$f_{xx}f_{yy} - f_{xy}^2 = (32)(6) > 0, \qquad \text{at } (x, y) = (\pm 2, 1).$$

  Since $f_{yy} > 0$, $f(x, y)$ has minima at $(\pm 2, 1)$.

11. Let

$$D = \left\{(x, y) : x^2 + y^2 \leq 1\right\}$$

be the unit disc and

$$f(x, y) = x^2 - 2x + y^2 + 2y + 1.$$

Find the global maximum and minimum of

$$f : D \to \mathbb{R}$$

At what points (x, y) in D does f attain its maximum and minimum?

Solution.

- The function is differentiable everywhere. A critical point (x, y) satisfies

  $$f_x = 2x - 2 = 0, \qquad 2y + 2 = 0$$

  which implies that $(x, y) = (1, -1)$. This point does not lie inside D, so f has no critical points inside D and it must attain its global maximum and minimum on the boundary of D.

- On the boundary $x^2 + y^2 = 1$, we can write

  $$x = \cos\theta, \quad y = \sin\theta$$

  where $0 \leq \theta \leq 2\pi$ is the polar angle of (x, y). Then

  $$f(\cos\theta, \sin\theta) = \cos^2\theta - 2\cos\theta + \sin^2\theta + 2\sin\theta + 1 = 2 - 2\cos\theta + 2\sin\theta.$$

- Since $f(\cos\theta, \sin\theta)$ is a differentiable function of θ, the maximum and minimum of f on the boundary are attained at a critical point where

  $$\frac{d}{d\theta} f(\cos\theta, \sin\theta) = 2\sin\theta + 2\cos\theta = 0.$$

  It follows that

  $$\tan\theta = -1,$$

  so $\theta = 3\pi/4$ or $\theta = 7\pi/4$.

- The second derivative of f along the boundary is

$$\frac{d^2}{d^2\theta} f(\cos\theta, \sin\theta) = 2\cos\theta - 2\sin\theta$$

We have

$$\cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}, \quad \sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}, \quad \cos\frac{7\pi}{4} = \frac{1}{\sqrt{2}}, \quad \sin\frac{7\pi}{4} = -\frac{1}{\sqrt{2}}$$

so

$$\frac{d^2 f}{d^2\theta} < 0 \quad \text{at } \theta = 3\pi/4, \qquad \frac{d^2 f}{d^2\theta} > 0 \quad \text{at } \theta = 7\pi/4.$$

Thus, f has a maximum value at $\theta = 3\pi/4$ and a minimum value at $\theta = 7\pi/4$.

- Evaluating the corresponding values of (x, y) and f, we find that f has a global maximum on D at

$$(x, y) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \qquad \text{where} \quad f(x, y) = 2 + 2\sqrt{2},$$

and a global minimum on D at

$$(x, y) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \qquad \text{where} \quad f(x, y) = 2 - 2\sqrt{2}.$$

12. Suppose that the material for the top and bottom of a rectangular box costs a dollars per square meter and the material for the four sides costs b dollars per square meter. Use the method of Lagrange multipliers to find the dimensions of a box of volume V cubic meters that minimizes the cost of the materials used to construct it. What is the minimal cost?

Solution.

- Let x, y be the lengths of the horizontal sides of the box and z the height of the box.
- The cost of the material for the box is

  $$C(x, y, z) = 2axy + 2bxz + 2byz.$$

  The volume of the box is xyz.
- Thus, we want to

  $$\text{minimize } C(x, y, z)$$

  subject to the constraint that

  $$g(x, y, z) = 0$$

  where

  $$g(x, y, z) = xyz - V.$$
- We have

  $$\begin{aligned} \nabla C &= (2ay + 2bz)\,\vec{i} + (2ax + 2bz)\,\vec{j} + (2bx + 2by)\,\vec{k}, \\ \nabla g &= yz\vec{i} + xz\vec{j} + xy\vec{k} \end{aligned}$$
- By the method of Lagrange multipliers, the equations for a critical point are $\nabla C = \lambda \nabla g$ and $g = 0$, or

  $$2ay + 2bz = \lambda yz, \quad 2ax + 2bz = \lambda xz, \quad 2bx + 2by = \lambda xy, \quad xyz = V.$$
- It follows from the first two equations that $x = y$, as one would expect by symmetry, and from the last equation that

  $$z = \frac{V}{x^2}.$$

Hence, assuming that $x > 0$, we find that

$$2ax + \frac{2bV}{x^2} = \frac{\lambda V}{x}, \qquad 4b = \lambda x$$

and, eliminating λ, we get

$$2ax = \frac{2bV}{x^2}.$$

It follows that

$$x = y = \left(\frac{b}{a}\right)^{1/3} V^{1/3}, \qquad z = \left(\frac{a}{b}\right)^{2/3} V^{1/3}.$$

- **Remark.** Note that a/b is a dimensionless ratio of costs per unit area and $V^{1/3}$ is a length, so these results have the correct dimensions (meters). If $a = b$, then $x = y = z$, which corresponds to the fact that the cube is the rectangular solid with minimal surface area that encloses a given volume. If $a > b$ then the box is taller than it is wide to reduce the amount of the more expensive material required for the top and bottom, while if $a < b$ then the box is shorter than it is wide to reduce the amount of material required for the sides.