1. Do the following sequences \( \{a_n\} \) converge or diverge as \( n \to \infty \)? Give reasons for your answer. If a sequence converges, find its limit.

(a) \( a_n = \frac{\cos n}{n} \);
(b) \( a_n = \frac{\sqrt{n}}{\ln n} \);
(c) \( a_n = \sqrt{n^2 + 1} - n \).

Solution.

- (a) We have
  \[
  -\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}.
  \]
  Since
  \[
  -\frac{1}{n} \to 0, \quad \frac{1}{n} \to 0 \quad \text{as} \quad n \to \infty,
  \]
  the ‘sandwich’ theorem implies that \( \frac{\cos n}{n} \to 0 \) as \( n \to \infty \). So this sequence converges to 0.

- (b) Since \( \sqrt{x} \to \infty \) and \( \ln x \to \infty \) as \( x \to \infty \), l’Hôpital’s rule implies that
  \[
  \lim_{n \to \infty} \frac{\sqrt{n}}{\ln n} = \lim_{x \to \infty} \frac{\sqrt{x}}{\ln x} = \lim_{x \to \infty} \frac{1/(2\sqrt{x})}{1/x} = \lim_{x \to \infty} \frac{\sqrt{x}}{2} = \infty.
  \]
  So this sequence diverges to \( \infty \).
(c) We have

\[
\lim_{n \to \infty} \left( \sqrt{n^2 + 1} - n \right) = \lim_{n \to \infty} \frac{(\sqrt{n^2 + 1} - n) (\sqrt{n^2 + 1} + n)}{\sqrt{n^2 + 1} + n} \\
= \lim_{n \to \infty} \frac{(n^2 + 1) - n^2}{\sqrt{n^2 + 1} + n} \\
= \lim_{n \to \infty} \frac{1}{\sqrt{n^2 + 1} + n} \\
= \lim_{n \to \infty} \frac{1}{n} \left( \frac{1}{\sqrt{1 + 1/n^2} + 1} \right) \\
= 0
\]

So this sequence converges to 0.
2. Do the following series converge absolutely, converge conditionally, or diverge? Give reasons for your answer.

\[(a) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}; \quad (b) \sum_{n=1}^{\infty} \frac{\sin n}{n^2}; \quad (c) \sum_{n=1}^{\infty} (-1)^n \sin n\]

**Solution.**

- (a) The series of absolute values,

\[\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}\]

is a \(p\)-series with \(p = 1/2\), which diverges since \(p < 1\). Thus, the series is not absolutely convergent. Since \(u_n = 1/\sqrt{n}\) is a decreasing positive sequence with \(u_n \to 0\) as \(n \to \infty\), the alternating series test implies that the series converges. So the series is conditionally convergent.

- (b) The series of absolute values

\[\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}\]

converges by the comparison test, since

\[0 \leq \frac{|\sin n|}{n^2} \leq \frac{1}{n^2}\]

and

\[\sum_{n=1}^{\infty} \frac{1}{n^2}\]

is a convergent \(p\)-series (with \(p = 2 > 1\)). So the series is absolutely convergent.

- (c) The limit

\[\lim_{n \to \infty} (-1)^n \sin n\]

\[3\]
doesn’t exist since the sequence oscillates as $n \to \infty$. For example, since $2\pi/3 > 2$, for every positive integer $k$, there are even integers $m$, $n$ with
\[
2\pi k + \frac{\pi}{6} < m < 2k\pi + \frac{5\pi}{6}, \quad 2\pi k + \frac{7\pi}{6} < n < 2k\pi + \frac{11\pi}{6}
\]
such that $\sin m > 1/2$ and $\sin n < -1/2$. So the series diverges by the $n$th term test.
3. Determine whether each of the following series converges or diverges and explain your answer:

(a) \[ \sum_{n=1}^{\infty} \frac{n + 4}{6n - 17}; \]
(b) \[ \sum_{n=1}^{\infty} \left( \frac{-4}{5} \right)^n ; \]
(c) \[ \sum_{n=2}^{\infty} \sqrt{\frac{n}{n^4 + 7}} ; \]
(d) \[ \sum_{n=1}^{\infty} \frac{5^{n+1}}{(2n)!}; \]
(e) \[ \sum_{n=1}^{\infty} \frac{1}{n \ln^2 n} ; \]
(f) \[ \sum_{n=3}^{\infty} \frac{(-1)^{n+1}}{n + 2\sqrt{n}} ; \]
(g) \[ \frac{1}{1^4} + \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} - \frac{1}{6^4} + \frac{1}{7^4} + \frac{1}{8^4} - \frac{1}{9^4} + \cdots ; \]
(h) \[ \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}; \]
(i) \[ \sum_{n=1}^{\infty} \left[ \tan(n) - \tan(n + 1) \right] . \]

Solution. Write each series as \( \sum a_n . \)

• (a) We have
  \[ \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n + 4}{6n - 17} = \lim_{n \to \infty} \frac{1 + 4/n}{6 - 17/n} = \frac{1}{6} . \]
  Since the limit of the terms \( a_n \) is nonzero, the series diverges by the \( n \)th term test.

• (b) This series is a geometric series with ratio \( r = -4/5 . \) Since \( |r| < 1 , \) the series converges. In fact, we have
  \[ \sum_{n=1}^{\infty} \left( \frac{-4}{5} \right)^n = -\frac{4}{5} \sum_{n=0}^{\infty} \left( \frac{-4}{5} \right)^n = -\frac{4}{5} \left( \frac{1}{1 - (-4/5)} \right) = -\frac{4}{5} \cdot \frac{5}{9} = -\frac{4}{9} . \]

• (c) We use the limit comparison test and compare with the \( p \)-series
  \[ \sum_{n=2}^{\infty} b_n , \quad b_n = \frac{1}{n^{3/2}} . \]
This series converges since \( p = \frac{3}{2} > 1 \). Moreover,

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\sqrt{n/(n^4 + 7)}}{1/n^{3/2}}
\]
\[
= \lim_{n \to \infty} \frac{n^{3/2} \sqrt{n}}{\sqrt{n^4 + 7}}
\]
\[
= \lim_{n \to \infty} \frac{1}{\sqrt{1 + 7/n^4}}
\]
\[
= 1.
\]

Since this limit is finite and \( \sum b_n \) converges, the limit comparison test implies that the series converges.

- **(d)** We use the ratio test. We have

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{5^{n+2}/(2n + 2)!}{5^{n+1}/(2n)!}
\]
\[
= \lim_{n \to \infty} \frac{5(2n)!}{(2n + 2)!}
\]
\[
= \lim_{n \to \infty} \frac{5}{(2n + 1) \cdot (2n + 2)}
\]
\[
= 0.
\]

Since this limit is less than 1, the ratio test implies that the series converges.

- **(e)** We use the integral test. Since \( 1/(x \ln^2 x) \) is a continuous, positive, decreasing function for \( x \geq 2 \), the series converges or diverges with the integral

\[
\int_2^\infty \frac{1}{x \ln^2 x} \, dx.
\]
Making the substitution \( u = \ln x \), with \( du = dx/x \), we find that

\[
\int_2^\infty \frac{1}{x \ln^2 x} \, dx = \int_{\ln 2}^{\infty} \frac{du}{u^2}
\]

\[
= \lim_{b \to \infty} \int_{\ln 2}^{b} \frac{du}{u^2}
\]

\[
= \lim_{b \to \infty} \left[ -\frac{1}{u} \right]_{\ln 2}^{b}
\]

\[
= \frac{1}{\ln 2}.
\]

Since this integral converges, the series converges.

- (f) The sequence

\[
u_n = \frac{1}{n + 2\sqrt{n}}
\]

is a positive, decreasing sequence such that \( u_n \to 0 \) as \( n \to \infty \). Therefore, the series \( \sum (-1)^{n+1} u_n \) converges by the alternating series test.

- (g) We have

\[
|a_n| = \frac{1}{n^4},
\]

so \( \sum |a_n| \) is a convergent \( p \)-series (with \( p = 4 > 1 \)). Therefore the series \( \sum a_n \) converges absolutely, and it converges.

- (h) We use the ratio test. We have

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{((n + 1)!)^2 / (2(n + 1))!}{(n!)^2 / (2n)!}
\]

\[
= \lim_{n \to \infty} \frac{((n + 1)!)^2 (2n)!}{(n!)^2 (2n + 2)!}
\]

\[
= \lim_{n \to \infty} \frac{(n + 1)^2}{(2n + 1) \cdot (2n + 2)}
\]

\[
= \lim_{n \to \infty} \frac{(1 + 1/n)^2}{(2 + 1/n) \cdot (2 + 2/n)}
\]

\[
= \frac{1}{4}.
\]

Since this limit is less than 1, the series converges by the ratio test.
(i) This series is a telescoping series. The $n$th partial sum is

$$s_n = \sum_{k=1}^{n} [\tan(k) - \tan(k + 1)]$$

$$= [\tan(1) - \tan(2)] + [\tan(2) - \tan(3)]$$
$$+ [\tan(3) - \tan(4)] + \cdots + [\tan(n) - \tan(n + 1)]$$

$$= \tan(1) - \tan(n + 1).$$

Since

$$\lim_{n \to \infty} \tan(n + 1)$$

does not exist, the series does not converge.
4. Are the following equalities true or false? Justify your answer.

(a) \[ 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \frac{1}{7^2} + \ldots = 1 + \frac{1}{3^2} - \frac{1}{2^2} + \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{4^2} + \frac{1}{9^2} + \frac{1}{11^2} - \frac{1}{6^2} + \ldots; \]

(b) \[ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \ldots = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \ldots; \]

Solution.

• (a) The series

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \frac{1}{7^2} + \ldots \]

converges absolutely since

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} \]

is a convergent \( p \)-series (with \( p = 2 > 1 \)). Therefore any rearrangement of the series converges to the same sum, and the equality is true.

• (b) The series

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \ldots \]

converges by the alternating series test, but it does not converge absolutely since

\[ \sum_{n=1}^{\infty} \frac{1}{n} \]

is the divergent harmonic series. Therefore a rearrangement of the series need not, in general, converge; moreover, if a rearrangement does converge, it need not converge to the same sum. So the sums in (b) need not be equal.
This leaves open the question of whether or not the particular rearrangements given in (b) converge to the same sum. The sums are, in fact, not equal, but it requires a trickier argument to show this. We give the details for completeness. The following discussion is optional and goes beyond what will be asked in the midterm.

Let
\[
s_{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{2n-1} - \frac{1}{2n},
\]
\[
r_{2n} = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \ldots
\]
\[
\cdots + \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n}
\]
denote suitable partial sums of the arrangements in (b). Then, looking at the terms included in \(s_{2n}\) and \(r_{2n}\), we see that
\[
r_{2n} = s_{2n} + e_n\tag{1}
\]
where
\[
e_n = \frac{1}{2n+1} + \frac{1}{2n+3} + \ldots + \frac{1}{4n-3} + \frac{1}{4n-1}.
\tag{2}
\]
There are \(n\) terms in this expression for \(e_n\) and each term is less than or equal to \(1/(2n+1)\) and greater than or equal to \(1/(4n-1)\). Thus,
\[
s_{2n} + n \cdot \left(\frac{1}{4n-1}\right) \leq r_{2n} \leq s_{2n} + n \cdot \left(\frac{1}{2n+1}\right)\tag{3}
\]

Let
\[
S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots,
\]
\[
R = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \ldots.
\]
The series for \(S\) converges by the alternating series test. By considering the partial sums of the series for \(R\) one can show in a similar way to the proof of the alternating series test that the series for \(R\) converges. Then we have
\[
S = \lim_{n \to \infty} s_{2n}, \quad R = \lim_{n \to \infty} r_{2n}.
\]
Taking the limit of (3) as \( n \to \infty \), we get

\[
S + \lim_{n \to \infty} \left( \frac{n}{4n-1} \right) \leq R \leq S + \lim_{n \to \infty} \left( \frac{n}{2n+1} \right)
\]

or

\[
S + \frac{1}{4} \leq R \leq S + \frac{1}{2}.
\]  \hspace{1cm} (4)

Thus, the two rearrangements cannot converge to the same sum.

• We can find \( R \) exactly in terms of \( S \) by observing that

\[
\frac{1}{2n+1} + \frac{1}{2n+3} + \cdots + \frac{1}{4n-1} = \frac{1}{2n} \left( \frac{1}{1 + (1/2n)} + \frac{1}{1 + (3/2n)} + \cdots + \frac{1}{2 - (1/2n)} \right)
\]

is a Riemann sum for the integral

\[
\frac{1}{2} \int_1^2 \frac{1}{x} \, dx.
\]

Therefore, if \( e_n \) is given by (2), we have

\[
\lim_{n \to \infty} e_n = \lim_{n \to \infty} \left( \frac{1}{2n+1} + \frac{1}{2n+3} + \cdots + \frac{1}{4n-3} + \frac{1}{4n-1} \right) = \frac{1}{2} \int_1^2 \frac{1}{x} \, dx = \frac{1}{2} \ln 2.
\]  \hspace{1cm} (5)

Taking the limit of (1) as \( n \to \infty \) and using (5), we get

\[
R = S + \frac{1}{2} \ln 2.
\]

Note that \( \ln 2/2 \approx 0.397 \) lies between 1/4 and 1/2 as required by (4).

• In fact, the sum of the alternating harmonic series is \( S = \ln 2 \), so we get the results

\[
1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots = \ln 2,
\]

\[
1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots = \frac{3}{2} \ln 2.
\]
5. State the definition for a sequence \( \{a_n\} \) to converge to a limit \( L \). If
\[
a_n = \frac{n^2 + 1}{n^2} \quad \text{for } n = 1, 2, 3, \ldots
\]
prove from the definition that
\[
\lim_{n \to \infty} a_n = 1.
\]

Solution.

• The definition of \( \lim_{n \to \infty} a_n = L \)
is that for every \( \epsilon > 0 \) there exists a number \( N \) such that
\[
|a_n - L| < \epsilon \quad \text{whenever } n > N.
\]

• Let \( \epsilon > 0 \). Choose \( N = 1/\sqrt{\epsilon} \). Then if \( n > N \),
\[
|a_n - 1| = \left| \frac{n^2 + 1}{n^2} - 1 \right|
= \left| \frac{1}{n^2} \right|
< \frac{1}{N^2}
< \epsilon.
\]

Hence,
\[
\lim_{n \to \infty} \frac{n^2 + 1}{n^2} = 1.
\]
Additional question. Does the series
\[ \sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}} \]
converge or diverge? Justify your answer.

Solution.

- The series converges. We’ll show this in two different ways.

- **Method 1.** The terms in the series are given by
  \[ a_n = f(n), \quad f(x) = \frac{1}{(\ln x)^{\ln x}}. \]
  The function \( f \) is continuous, positive and decreasing, so by the integral test the series converges if and only if the improper Riemann integral
  \[ I = \int_{a}^{\infty} \frac{1}{(\ln x)^{\ln x}} \, dx \]
  converges for some \( a \geq 2 \). Using the substitution
  \[ x = e^u, \quad \ln x = u, \quad dx = e^u du \]
  we get that
  \[ I = \int_{c}^{\infty} \frac{e^u}{u^u} du = \int_{c}^{\infty} \left( \frac{e}{u} \right)^u du, \quad c = e^a. \]
  Let \( c = e^2 \) (any other choice \( c > e \) would also work). If \( c \leq u < \infty \), then \( e/u \leq e/c = 1/e \), so
  \[ \int_{c}^{\infty} \left( \frac{e}{u} \right)^u du \leq \int_{c}^{\infty} e^{-u} du = \lim_{b \to \infty} \int_{c}^{b} e^{-u} du = \lim_{b \to \infty} [e^{-c} - e^{-b}] = e^{-c}. \]
  Hence, the integral converges, so the series converges.

- **Method 2.** For any \( x > 0 \), we have \( x = e^{\ln x} \), so
  \[ \ln n = e^{\ln \ln n}. \]
It follows that
\[(\ln n)^{\ln n} = (e^{\ln \ln n})^{\ln n} = e^{\ln n \cdot \ln \ln n} = (e^{\ln n})^{\ln \ln n} = n^{\ln \ln n}.\]

Since \(\ln \ln n \to \infty\) as \(n \to \infty\), albeit very slowly, there exists \(N > 0\) such that \(\ln \ln n \geq 2\) for all \(n > N\). (In fact, we can take \(N = e^{e^2}\).)

Then for \(n > N\) we have
\[
0 \leq \frac{1}{(\ln n)^{\ln n}} = \frac{1}{n^{\ln \ln n}} \leq \frac{1}{n^2}.
\]

Since \(\sum \frac{1}{n^2}\) is a convergent \(p\)-series, the positive series \(\sum \frac{1}{(\ln n)^{\ln n}}\) converges by the comparison test.