Calculus

Math 21C, Spring 2018

Solutions to Sample Questions: Midterm I

1. Do the following sequences $\{a_n\}$ converge or diverge as $n \to \infty$? Give reasons for your answer. If a sequence converges, find its limit.

$$\text{(a)} \quad a_n = \frac{\cos n}{n}; \qquad \text{(b)} \quad a_n = \frac{\sqrt{n}}{\ln n}; \qquad \text{(c)} \quad a_n = \sqrt{n^2+1} - n.$$

Solution.

- (a) We have
$$-\frac{1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}.$$
Since
$$-\frac{1}{n} \to 0, \quad \frac{1}{n} \to 0 \qquad \text{as } n \to \infty,$$
the 'sandwich' theorem implies that $\cos n/n \to 0$ as $n \to \infty$. So this sequence converges to 0.

- (b) Since $\sqrt{x} \to \infty$ and $\ln x \to \infty$ as $x \to \infty$, l'Hôspital's rule implies that
$$\begin{aligned} \lim_{n\to\infty} \frac{\sqrt{n}}{\ln n} &= \lim_{x\to\infty} \frac{\sqrt{x}}{\ln x} \\ &= \lim_{x\to\infty} \frac{1/(2\sqrt{x})}{1/x} \\ &= \lim_{x\to\infty} \frac{\sqrt{x}}{2} \\ &= \infty. \end{aligned}$$
So this sequence diverges to ∞.

- (c) We have

$$\begin{aligned}
\lim_{n\to\infty}\left(\sqrt{n^2+1}-n\right) &= \lim_{n\to\infty}\frac{\left(\sqrt{n^2+1}-n\right)\left(\sqrt{n^2+1}+n\right)}{\sqrt{n^2+1}+n}\\
&= \lim_{n\to\infty}\frac{(n^2+1)-n^2}{\sqrt{n^2+1}+n}\\
&= \lim_{n\to\infty}\frac{1}{\sqrt{n^2+1}+n}\\
&= \lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{\sqrt{1+1/n^2}+1}\right)\\
&= 0
\end{aligned}$$

So this sequence converges to 0.

2. Do the following series converge absolutely, converge conditionally, or diverge? Give reasons for your answer.

$$\text{(a)}\quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}; \qquad \text{(b)}\quad \sum_{n=1}^{\infty} \frac{\sin n}{n^2}; \qquad \text{(c)}\quad \sum_{n=1}^{\infty} (-1)^n \sin n$$

Solution.

- (a) The series of absolute values,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

  is a p-series with $p = 1/2$, which diverges since $p < 1$. Thus, the series is not absolutely convergent. Since $u_n = 1/\sqrt{n}$ is a decreasing positive sequence with $u_n \to 0$ as $n \to \infty$, the alternating series test implies that the series converges. So the series is conditionally convergent.

- (b) The series of absolute values

$$\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$$

  converges by the comparison test, since

$$0 \le \frac{|\sin n|}{n^2} \le \frac{1}{n^2}$$

  and

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

  is a convergent p-series (with $p = 2 > 1$). So the series is absolutely convergent.

- (c) The limit

$$\lim_{n\to\infty} (-1)^n \sin n$$

doesn't exist since the sequence oscillates as $n \to \infty$. For example, since $2\pi/3 > 2$, for every positive integer k, there are even integers m, n with

$$2\pi k + \frac{\pi}{6} < m < 2k\pi + \frac{5\pi}{6}, \qquad 2\pi k + \frac{7\pi}{6} < n < 2\pi k + \frac{11\pi}{6}$$

such that $\sin m > 1/2$ and $\sin n < -1/2$. So the series diverges by the nth term test.

3. Determine whether each of the following series converges or diverges and explain your answer:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n+4}{6n-17}$; (b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{-4}{5}\right)^{n}$; (c) $\displaystyle\sum_{n=2}^{\infty} \sqrt{\frac{n}{n^4+7}}$;

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{5^{n+1}}{(2n)!}$; (e) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\ln^2 n}$; (f) $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^{n+1}}{n+2\sqrt{n}}$;

(g) $\displaystyle\frac{1}{1^4}+\frac{1}{2^4}-\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}-\frac{1}{6^4}+\frac{1}{7^4}+\frac{1}{8^4}-\frac{1}{9^4}+\cdots$;

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$; (i) $\displaystyle\sum_{n=1}^{\infty} \left[\tan(n)-\tan(n+1)\right]$.

Solution. Write each series as $\sum a_n$.

- (a) We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+4}{6n-17} = \lim_{n\to\infty} \frac{1+4/n}{6-17/n} = \frac{1}{6}.$$

  Since the limit of the terms a_n is nonzero, the series diverges by the nth term test.

- (b) This series is a geometric series with ratio $r=-4/5$. Since $|r|<1$, the series converges. In fact, we have

$$\sum_{n=1}^{\infty} \left(\frac{-4}{5}\right)^{n} = -\frac{4}{5}\sum_{n=0}^{\infty} \left(\frac{-4}{5}\right)^{n} = -\frac{4}{5}\left(\frac{1}{1-(-4/5)}\right) = -\frac{4}{5}\cdot\frac{5}{9} = -\frac{4}{9}.$$

- (c) We use the limit comparison test and compare with the p-series

$$\sum_{n=2}^{\infty} b_n, \qquad b_n = \frac{1}{n^{3/2}}.$$

This series converges since $p = 3/2 > 1$. Moreover,

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\sqrt{n/(n^4+7)}}{1/n^{3/2}} \\
&= \lim_{n\to\infty} \frac{n^{3/2}\sqrt{n}}{\sqrt{n^4+7}} \\
&= \lim_{n\to\infty} \frac{1}{\sqrt{1+7/n^4}} \\
&= 1.
\end{aligned}$$

Since this limit is finite and $\sum b_n$ converges, the limit comparison test implies that the series converges.

- (d) We use the ratio test. We have

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{5^{n+2}/(2n+2)!}{5^{n+1}/(2n)!} \\
&= \lim_{n\to\infty} \frac{5(2n)!}{(2n+2)!} \\
&= \lim_{n\to\infty} \frac{5}{(2n+1)\cdot(2n+2)} \\
&= 0.
\end{aligned}$$

Since this limit is less than 1, the ratio test implies that the series converges.

- (e) We use the integral test. Since $1/(x\ln^2 x)$ is a continuous, positive, decreasing function for $x \geq 2$, the series converges or diverges with the integral

$$\int_2^\infty \frac{1}{x\ln^2 x}\, dx.$$

Making the substitution $u = \ln x$, with $du = dx/x$, we find that

$$\begin{aligned}\int_2^\infty \frac{1}{x\ln^2 x}\,dx &= \int_{\ln 2}^\infty \frac{du}{u^2} \\ &= \lim_{b\to\infty}\int_{\ln 2}^b \frac{du}{u^2} \\ &= \lim_{b\to\infty}\left[-\frac{1}{u}\right]_{\ln 2}^b \\ &= \frac{1}{\ln 2}.\end{aligned}$$

Since this integral converges, the series converges.

- (f) The sequence
$$u_n = \frac{1}{n + 2\sqrt{n}}$$
is a positive, decreasing sequence such that $u_n \to 0$ as $n \to \infty$. Therefore, the series $\sum (-1)^{n+1} u_n$ converges by the alternating series test.

- (g) We have
$$|a_n| = \frac{1}{n^4},$$
so $\sum |a_n|$ is a convergent p-series (with $p = 4 > 1$). Therefore the series $\sum a_n$ converges absolutely, and it converges.

- (h) We use the ratio test. We have
$$\begin{aligned}\lim_{n\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{n\to\infty}\frac{\left((n+1)!\right)^2/(2(n+1))!}{(n!)^2/(2n)!} \\ &= \lim_{n\to\infty}\frac{\left((n+1)!\right)^2(2n)!}{(n!)^2\,(2n+2)!} \\ &= \lim_{n\to\infty}\frac{(n+1)^2}{(2n+1)\cdot(2n+2)} \\ &= \lim_{n\to\infty}\frac{(1+1/n)^2}{(2+1/n)\cdot(2+2/n)} \\ &= \frac{1}{4}.\end{aligned}$$
Since this limit is less than 1, the series converges by the ratio test.

- (i) This series is a telescoping series. The nth partial sum is

$$\begin{aligned} s_n &= \sum_{k=1}^{n} [\tan(k) - \tan(k+1)] \\ &= [\tan(1) - \tan(2)] + [\tan(2) - \tan(3)] \\ &\qquad + [\tan(3) - \tan(4)] + \cdots + [\tan(n) - \tan(n+1)] \\ &= \tan(1) - \tan(n+1). \end{aligned}$$

  Since
$$\lim_{n\to\infty} \tan(n+1)$$
  does not exist, the series does not converge.

4. Are the following equalities true or false? Justify your answer.

(a) $$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\frac{1}{6^2}+\frac{1}{7^2}+\dots$$
$$=1+\frac{1}{3^2}-\frac{1}{2^2}+\frac{1}{5^2}+\frac{1}{7^2}-\frac{1}{4^2}+\frac{1}{9^2}+\frac{1}{11^2}-\frac{1}{6^2}+\dots;$$

(b) $$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}+\dots$$
$$=1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\dots;$$

Solution.

- (a) The series

  $$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\frac{1}{6^2}+\frac{1}{7^2}+\dots$$

  converges absolutely since

  $$\sum_{n=1}^{\infty}\frac{1}{n^2}$$

  is a convergent p-series (with $p=2>1$). Therefore any rearrangement of the series converges to the same sum, and the equality is true.

- (b) The series

  $$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}+\dots$$

  converges by the alternating series test, but it does not converge absolutely since

  $$\sum_{n=1}^{\infty}\frac{1}{n}$$

  is the divergent harmonic series. Therefore a rearrangement of the series need not, in general, converge; moreover, if a rearrangement does converge, it need not converge to the same sum. So the sums in (b) need not be equal.

- This leaves open the question of whether or not the particular rearrangements given in (b) converge to the same sum. The sums are, in fact, not equal, but it requires a trickier argument to show this. We give the details for completeness. The following discussion is optional and goes beyond what will be asked in the midterm.

- Let

$$
\begin{aligned}
s_{2n} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}, \\
r_{2n} &= 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \ldots \\
&\qquad \cdots + \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n}
\end{aligned}
$$

  denote suitable partial sums of the arrangements in (b). Then, looking at the terms included in s_{2n} and r_{2n}, we see that

$$
r_{2n} = s_{2n} + e_n \tag{1}
$$

  where

$$
e_n = \frac{1}{2n+1} + \frac{1}{2n+3} + \cdots + \frac{1}{4n-3} + \frac{1}{4n-1}. \tag{2}
$$

  There are n terms in this expression for e_n and each term is less than or equal to $1/(2n+1)$ and greater than or equal to $1/(4n-1)$. Thus,

$$
s_{2n} + n \cdot \left(\frac{1}{4n-1} \right) \leq r_{2n} \leq s_{2n} + n \cdot \left(\frac{1}{2n+1} \right). \tag{3}
$$

- Let

$$
\begin{aligned}
S &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots, \\
R &= 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \ldots.
\end{aligned}
$$

  The series for S converges by the alternating series test. By considering the partial sums of the series for R one can show in a similar way to the proof of the alternating series test that the series for R converges. Then we have

$$
S = \lim_{n\to\infty} s_{2n}, \qquad R = \lim_{n\to\infty} r_{2n}.
$$

Taking the limit of (3) as $n \to \infty$, we get

$$S + \lim_{n\to\infty} \left(\frac{n}{4n-1} \right) \le R \le S + \lim_{n\to\infty} \left(\frac{n}{2n+1} \right)$$

or

$$S + \frac{1}{4} \le R \le S + \frac{1}{2}. \tag{4}$$

Thus, the two rearrangements cannot converge to the same sum.

- We can find R exactly in terms of S by observing that

$$\begin{aligned} &\frac{1}{2n+1} + \frac{1}{2n+3} + \cdots + \frac{1}{4n-1} \\ &\qquad = \frac{1}{2n} \left(\frac{1}{1+(1/2n)} + \frac{1}{1+(3/2n)} + \cdots + \frac{1}{2-(1/2n)} \right) \end{aligned}$$

is a Riemann sum for the integral

$$\frac{1}{2} \int_1^2 \frac{1}{x}\, dx.$$

Therefore, if e_n is given by (2), we have

$$\begin{aligned} \lim_{n\to\infty} e_n &= \lim_{n\to\infty} \left(\frac{1}{2n+1} + \frac{1}{2n+3} + \cdots + \frac{1}{4n-3} + \frac{1}{4n-1} \right) \\ &= \frac{1}{2} \int_1^2 \frac{1}{x}\, dx \qquad\qquad (5) \\ &= \frac{1}{2} \ln 2. \end{aligned}$$

Taking the limit of (1) as $n \to \infty$ and using (5), we get

$$R = S + \frac{1}{2} \ln 2.$$

Note that $\ln 2/2 \approx 0.397$ lies between $1/4$ and $1/2$ as required by (4).

- In fact, the sum of the alternating harmonic series is $S = \ln 2$, so we get the results

$$\begin{aligned} &1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots = \ln 2, \\ &1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots = \frac{3}{2} \ln 2. \end{aligned}$$

5. State the definition for a sequence $\{a_n\}$ to converge to a limit L. If

$$a_n = \frac{n^2+1}{n^2} \qquad \text{for } n = 1, 2, 3, \ldots$$

prove *from the definition* that

$$\lim_{n\to\infty} a_n = 1.$$

Solution.

- The definition of

  $$\lim_{n\to\infty} a_n = L$$

  is that for every $\epsilon > 0$ there exists a number N such that

  $$|a_n - L| < \epsilon \qquad \text{whenever } n > N.$$

- Let $\epsilon > 0$. Choose $N = 1/\sqrt{\epsilon}$. Then if $n > N$,

  $$\begin{aligned} |a_n - 1| &= \left|\frac{n^2+1}{n^2} - 1\right| \\ &= \left|\frac{1}{n^2}\right| \\ &< \frac{1}{N^2} \\ &< \epsilon. \end{aligned}$$

  Hence,

  $$\lim_{n\to\infty} \frac{n^2+1}{n^2} = 1.$$

Additional question. Does the series

$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$$

converge or diverge? Justify your answer.

Solution.

- The series converges. We'll show this in two different ways.
- *Method 1.* The terms in the series are given by

  $$a_n = f(n), \qquad f(x) = \frac{1}{(\ln x)^{\ln x}}.$$

  The function f is continuous, positive and decreasing, so by the integral test the series converges if and only if the improper Riemann integral

  $$I = \int_a^{\infty} \frac{1}{(\ln x)^{\ln x}}\, dx$$

  converges for some $a \geq 2$. Using the substitution

  $$x = e^u, \qquad \ln x = u, \qquad dx = e^u du$$

  we get that

  $$I = \int_c^{\infty} \frac{e^u}{u^u}\, du = \int_c^{\infty} \left(\frac{e}{u}\right)^u du, \qquad c = e^a.$$

  Let $c = e^2$ (any other choice $c > e$ would also work). If $c \leq u < \infty$, then $e/u \leq e/c = 1/e$, so

  $$\int_c^{\infty} \left(\frac{e}{u}\right)^u du \leq \int_c^{\infty} e^{-u}\, du = \lim_{b \to \infty} \int_c^b e^{-u}\, du = \lim_{b \to \infty} \left[e^{-c} - e^{-b}\right] = e^{-c}.$$

  Hence, the integral converges, so the series converges.
- *Method 2.* For any $x > 0$, we have $x = e^{\ln x}$, so

  $$\ln n = e^{\ln \ln n}.$$

It follows that

$$(\ln n)^{\ln n} = \left(e^{\ln\ln n}\right)^{\ln n} = e^{\ln n \cdot \ln\ln n} = \left(e^{\ln n}\right)^{\ln\ln n} = n^{\ln\ln n}.$$

Since $\ln\ln n \to \infty$ as $n \to \infty$, albeit *very* slowly, there exists $N > 0$ such that $\ln\ln n \geq 2$ for all $n > N$. (In fact, we can take $N = e^{e^2}$.) Then for $n > N$ we have

$$0 \leq \frac{1}{(\ln n)^{\ln n}} = \frac{1}{n^{\ln\ln n}} \leq \frac{1}{n^2}.$$

Since $\sum \frac{1}{n^2}$ is a convergent p-series, the positive series $\sum \frac{1}{(\ln n)^{\ln n}}$ converges by the comparison test.