Calculus: Final Solutions
Math 21D, Fall 2019

1. [20 pts] Let R be the triangular region in the (x, y)-plane with vertices $(0, 0)$, $(2, 1)$, $(3, 0)$. Sketch R and evaluate the double integral

$$\iint_R x \, dxdy.$$

Solution.

- The region R can be described by $0 \leq y \leq 1$, $2y \leq x \leq 3 - y$, so

$$\begin{aligned}
\iint_R x \, dxdy &= \int_0^1 \int_{2y}^{3-y} x \, dxdy \\
&= \frac{1}{2} \int_0^1 \left[(3-y)^2 - 4y^2\right] dy \\
&= \frac{1}{2} \int_0^1 \left[9 - 6y - 3y^2\right] dy \\
&= \frac{1}{2} \left[9y - 3y^2 - y^3\right]_0^1 \\
&= \frac{5}{2}.
\end{aligned}$$

2. [20 pts] Let D be the volume in (x, y, z)-space with $x^2 + y^2 \leq 1$ and $1 \leq z \leq 3$. Evaluate the triple integral

$$\iiint_D \left(x^2 + y^2 + z^2\right) \, dxdydz.$$

Solution.

- We use cylindrical coordinates (r, θ, z). The volume D is described by $r^2 \leq 1$ and $1 \leq z \leq 3$. In addition, $dV = r dr d\theta dz$ and $x^2 + y^2 = r^2$, so

$$\begin{aligned}
\iiint_D \left(x^2 + y^2 + z^2\right) \, dxdydz &= \int_1^3 \int_0^{2\pi} \int_0^1 \left(r^2 + z^2\right) \, r dr d\theta dz \\
&= 2\pi \int_1^3 \left[\frac{1}{4} r^4 + \frac{1}{2} r^2 z^2\right]_0^1 \, dz \\
&= 2\pi \int_1^3 \left(\frac{1}{4} + \frac{1}{2} z^2\right) \, dz \\
&= 2\pi \left[\frac{1}{4} z + \frac{1}{6} z^3\right]_1^3 \\
&= \frac{29\pi}{3}.
\end{aligned}$$

3. [20 pts] (a) Suppose that $a, b > 0$ are constants and define a transformation from the (u, v)-plane to the (x, y) plane by $x = au$, $y = bv$. What is the Jacobian of this transformation?

(b) Let R be elliptical region in the (x, y)-plane given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1.$$

What is the corresponding region G in the (u, v)-plane?

(c) Use the change of variables in (a) to evaluate the double integral

$$\iint_R \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^{1/2} dxdy$$

Solution.

- (a) The Jacobian is

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$$

- (b) The region G in the (u, v) plane is the unit disc $u^2 + v^2 \leq 1$.

- (c) We have

$$\iint_R \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^{1/2} dxdy = ab \iint_G \left(u^2 + v^2\right)^{1/2} dudv.$$

  To evaluate this integral, we use polar coordinates (r, θ) in the (u, v)-plane, which gives

$$\iint_G \left(u^2 + v^2\right)^{1/2} dudv = \int_0^{2\pi} \int_0^1 r \cdot r dr d\theta = 2\pi \left[\frac{1}{3} r^3\right]_0^1 = \frac{2\pi}{3},$$

  so

$$\iint_R \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^{1/2} dxdy = \frac{2\pi ab}{3}.$$

4. [20 pts] (a) State Green's theorem.

(b) Let C be the closed rectangular curve in the (x, y)-plane, oriented counterclockwise, that consists of the line segments from $(0, 0)$ to $(2, 0)$, from $(2, 0)$ to $(2, 3)$, from $(2, 3)$ to $(0, 3)$, and from $(0, 3)$ to $(0, 0)$. Use Green's theorem to evaluate the line integral

$$\oint_C xe^{y/3}\,dx + y^2 e^{x/2}\,dy.$$

Solution.

- (a) Green's theorem: If $M(x, y)$ and $N(x, y)$ are continuously differentiable functions defined on a bounded region R in the (x, y)-plane with positively oriented, piecewise smooth boundary curve C, then

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \oint_C Mdx + Ndy.$$

- (b) The region R enclosed by C is the rectangle $0 \le x \le 2$, $0 \le y \le 3$. Taking $M(x, y) = xe^{y/3}$ and $N(x, y) = y^2 e^{x/2}$ in Green's theorem, we get that

$$\begin{aligned}
\oint_C xe^{y/3}\,dx + y^2 e^{x/2}\,dy &= \iint_R \frac{\partial}{\partial x}\left(y^2 e^{x/2}\right) - \frac{\partial}{\partial y}\left(xe^{y/3}\right)\,dxdy \\
&= \int_0^3 \int_0^2 \left(\frac{1}{2}y^2 e^{x/2} - \frac{1}{3}xe^{y/3}\right) dxdy \\
&= \int_0^2 \frac{1}{2}e^{x/2}\,dx \cdot \int_0^3 y^2\,dy - \int_0^2 x\,dx \cdot \int_0^3 \frac{1}{3}e^{y/3}\,dy \\
&= \left[e^{x/2}\right]_0^2 \cdot \left[\frac{1}{3}y^3\right]_0^3 - \left[\frac{1}{2}x^2\right]_0^2 \cdot \left[e^{y/3}\right]_0^3 \\
&= 9(e-1) - 2(e-1) \\
&= 7(e-1).
\end{aligned}$$

5. [20 pts] A particle moves along a helical path

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + (bt)\mathbf{k}, \qquad 0 \le t \le 2\pi,$$

where $a, b > 0$ are constants, in a force-field

$$\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j} + z\mathbf{k}.$$

Compute the work done by the force on the particle.

Solution.

- The work W is given by the line integral

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt.$$

  We have

$$\begin{aligned} \mathbf{F}(\mathbf{r}(t)) &= -a\sin t\mathbf{i} + a\cos t\mathbf{j} + bt\mathbf{k}, \\ \mathbf{r}'(t) &= -a\sin t\mathbf{i} + a\cos t\mathbf{j} + b\mathbf{k}, \\ \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= a^2\sin^2 t + a^2\cos^2 t + b^2 t = a^2 + b^2 t. \end{aligned}$$

  It follows that

$$W = \int_0^{2\pi} a^2 + b^2 t\, dt = \left[a^2 t + \frac{1}{2}b^2 t^2\right]_0^{2\pi} = 2\pi a^2 + 2\pi^2 b^2.$$

6. [20 pts] Let

$$\mathbf{F}(x,y,z) = [\sin(x+y)+\sin(y+z)]\,\mathbf{i} + [\sin(x+y)+x\cos(y+z)]\,\mathbf{j} \\ + [x\cos(y+z)+\cos z]\,\mathbf{k}$$

(a) Find a potential $f(x,y,z)$ such that $\mathbf{F} = \nabla f$.
(b) What is $\nabla \times \mathbf{F}$?

Solution.

- (a) If $\nabla f = \mathbf{F}$, then by equating x-components, we must have

  $$f_x(x,y,z) = \sin(x+y) + \sin(y+z),$$

  and an integration with respect to x implies that

  $$f(x,y,z) = -\cos(x+y) + x\sin(y+z) + a(y,z)$$

  where $a(y,z)$ is a function of integration.

- Equating y-components in $\nabla f = \mathbf{F}$, and using this expression for f, we get that

  $$\sin(x+y) + x\cos(y+z) + a_y(y,z) = \sin(x+y) + x\cos(y+z),$$

  so $a_y = 0$ and $a = a(z)$ depends only on z.

- Finally, equating z-components and using the previous expression for f, we have

  $$x\cos(y+z) + a'(z) = x\cos(y+z) + \cos z,$$

  so $a'(z) = \cos z$ and $a(z) = \sin z + C$, where C is a constant of integration. Hence $\mathbf{F} = \nabla f$ where

  $$f(x,y,z) = -\cos(x+y) + x\sin(y+z) + \sin z + C.$$

  (b) We have $\nabla \times \mathbf{F} = 0$, since the curl of a gradient is always 0.

7. [20 pts] Let S be the sphere $x^2+y^2+z^2=a^2$ of radius $a>0$ and $\mathbf{F}=z\mathbf{k}$.
(a) What is the unit outward normal $\mathbf{n}$ to S? Sketch a picture of S, the outward normal $\mathbf{n}$, and the vector field $\mathbf{F}$ on S. Do you expect the outward flux of $\mathbf{F}$ across S to be positive, negative, or zero?
(b) Use a parametrization of S by spherical polar angles (ϕ, θ) to evaluate the outward flux

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, d\sigma$$

of $\mathbf{F}$ across S. HINT. You can use the surface element $d\sigma = a^2 \sin\phi\, d\phi d\theta$.

Solution.

- (a) The surface S is a level surface of the function $x^2+y^2+z^2$, so a normal vector is

$$\nabla(x^2+y^2+z^2) = 2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}.$$

  Dividing by the magnitude of this vector, we get the unit normal vector on S

$$\mathbf{n} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{\sqrt{x^2+y^2+z^2}} = \frac{1}{a}\left(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}\right)$$

  which is pointing outwards.

- The outward normal component of $\mathbf{F}$ is positive (and zero for $z=0$), so the outward flux of $\mathbf{F}$ across S is positive.

- (b) We have $z = a\cos\phi$ on S and

$$\mathbf{F}\cdot\mathbf{n} = \frac{z^2}{a} = a\cos^2\phi.$$

  Using a substitution $u=\cos\phi$, it follows that

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\, d\sigma &= \int_0^{2\pi}\int_0^{\pi} a\cos^2\phi\cdot a^2\sin\phi d\phi d\theta \\ &= 2\pi a^3\int_0^{\pi}\cos^2\phi\sin\phi d\phi \\ &= 2\pi a^3\int_{-1}^{1} u^2 du \\ &= \frac{4\pi a^3}{3}.\end{aligned}$$

8. [20 pts] Suppose that $\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ is a twice continuously differentiable vector field.

(a) Define $\nabla \cdot \mathbf{F}$.

(b) Define $\nabla \times \mathbf{F}$. Write out the components of $\nabla \times \mathbf{F}$ explicitly.

(c) Show that $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

(d) Suppose that D is a volume in space with boundary surface S. What does the divergence theorem tell you about the flux integral $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma$?

Solution.

- (a) We have $\nabla \cdot \mathbf{F} = M_x + N_y + P_z$.
- (b) We have

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & P \end{vmatrix} \\
&= (P_y - N_z)\mathbf{i} + (M_z - P_x)\mathbf{j} + (N_x - M_y)\mathbf{k}
\end{aligned}$$

- (c) We compute that

$$\begin{aligned}
\nabla \cdot (\nabla \times \mathbf{F}) &= \frac{\partial}{\partial x}(P_y - N_z) + \frac{\partial}{\partial y}(M_z - P_x) + \frac{\partial}{\partial z}(N_x - M_y) \\
&= P_{yx} - N_{zx} + M_{zy} - P_{xy} + N_{xz} - M_{yz} \\
&= 0,
\end{aligned}$$

  where all the terms cancel because of the equality of mixed partial derivatives for twice continuously differentiable functions ($P_{yx} = P_{xy}$ and so on).

- (d) Using the divergence theorem and the result in (c), we have

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot (\nabla \times \mathbf{F})\, dV = 0,$$

  meaning that if $\nabla \times \mathbf{F}$ is defined and continuous in a bounded region D enclosed by a (piecewise smooth) surface S, then the flux of $\nabla \times \mathbf{F}$ through S is zero.

9. [20 pts] (a) State Stokes theorem.
(b) Suppose that the surface S_1 is the upper hemisphere $x^2+y^2+z^2 = 1$, $z \geq 0$ and S_2 is the lower hemisphere $x^2+y^2+z^2 = 1$, $z \leq 0$, oriented by the outward unit normal $\mathbf{n}$ on $x^2+y^2+z^2 = 1$. If $\mathbf{F}(x, y, z)$ is a (continuously differentiable) vector field, what does Stokes theorem tell you about the relationship between the flux integrals $\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma$ and $\iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma$?
(c) Does the relationship in (b) hold for flux integrals $\iint_{S_1} \mathbf{G} \cdot \mathbf{n}\, d\sigma$ and $\iint_{S_2} \mathbf{G} \cdot \mathbf{n}\, d\sigma$ of general vector fields $\mathbf{G}$?

Solution.

- (a) If S is a piecewise smooth, oriented surface with piecewise smooth, positively oriented boundary curve C and $\mathbf{F}$ is a continuously differentiable vector field on S, then

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \oint_C \mathbf{F} \cdot \mathbf{T}\, ds,$$

  meaning that the flux of $\nabla \times \mathbf{F}$ across S is equal to the circulation of $\mathbf{F}$ around C.

- (b) The boundary curve of S_1 is the unit circle $C_1 : x^2 + y^2 = 1$ in the (x, y)-plane. By the right-hand rule, the positive orientation of C_1 with respect to the unit normal on S_1, which is upward, is counter-clockwise. By Stokes theorem,

$$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \oint_{C_1} \mathbf{F} \cdot \mathbf{T}\, ds.$$

- The boundary curve of S_2 is the unit circle $C_2 : x^2 + y^2 = 1$ in the (x, y)-plane. By the right-hand rule, the positive orientation of C_2 with respect to the unit normal on S_2, which is downward, is clockwise. By Stokes theorem,

$$\iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \oint_{C_2} \mathbf{F} \cdot \mathbf{T}\, ds.$$

- Since C_1, C_2 are the same curve with opposite orientations, it follows that $\oint_{C_1} = -\oint_{C_2}$, so

$$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = -\iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma.$$

- (c) There is no such relation for general vector fields $\mathbf{G}$. In fact, by choosing $\mathbf{G}$ in one way on S_1 and another way on S_2, we could make the flux integrals $\iint_{S_1} \mathbf{G} \cdot \mathbf{n}\, d\sigma$ and $\iint_{S_2} \mathbf{G} \cdot \mathbf{n}\, d\sigma$ have any values we want.

Remark. If $S = S_1 + S_2$ is the entire unit sphere $x^2 + y^2 + z^2 = 1$, with outward unit normal $\mathbf{n}$, then

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma + \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = 0,$$

so this result is consistent with the result in 8(d) for the flux of $\nabla \times \mathbf{F}$ through a closed surface S.