CALCULUS
Math 21D, Fall 2019
Solutions: Sample Questions Midterm II

1. (a) Write down a parametric equation for the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a, b > 0$ are constants.
(b) Use the parametrization in (a) to obtain an integral for the arclength of the ellipse. Do not attempt to evaluate the integral.

Solution.

- (a) A natural parametrization is

$$x = a\cos t, \quad y = b\sin t, \qquad 0 \le t \le 2\pi,$$

  with position vector $\mathbf{r}(t) = (a\cos t)\mathbf{i} + (b\sin t)\mathbf{j}$.

- (b) The arclength L is

$$\begin{aligned} L &= \int_0^{2\pi} |\mathbf{r}'(t)|\,dt \\ &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\ &= \int_0^{2\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\,dt. \end{aligned}$$

2. Let C be a curve parametrized by $-1 < s < 1$ with position vector

$$\mathbf{r}(s) = \frac{1}{3}(1+s)^{3/2}\mathbf{i} + \frac{1}{3}(1-s)^{3/2}\mathbf{j} + \frac{1}{\sqrt{2}}s\mathbf{k}.$$

(a) Show that s is an arclength parameter for the curve.
(b) Find the unit tangent vector $\mathbf{T}$.
(c) Find the unit normal vector $\mathbf{N}$.
(d) Find the binormal vector $\mathbf{B} = \mathbf{T} \times \mathbf{N}$.
(e) Find the curvature κ.

Solution.

- (a) The parameter s is an arclength parameter if $|\mathbf{r}'(s)| = 1$. We compute that

$$\mathbf{r}'(s) = \frac{1}{2}(1+s)^{1/2}\mathbf{i} - \frac{1}{2}(1-s)^{1/2}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k},$$
$$|\mathbf{r}'(s)| = \sqrt{\frac{1}{4}(1+s) + \frac{1}{4}(1-s) + \frac{1}{2}} = 1.$$

- (b) Since $\mathbf{r}'$ is a unit vector, the unit tangent vector is $\mathbf{T} = \mathbf{r}'$, or

$$\mathbf{T}(s) = \frac{1}{2}(1+s)^{1/2}\mathbf{i} - \frac{1}{2}(1-s)^{1/2}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}.$$

- (c) The unit normal vector is

$$\mathbf{N} = \frac{\mathbf{T}'}{|\mathbf{T}'|}.$$

  We have

$$\mathbf{T}'(s) = \frac{1}{4}(1+s)^{-1/2}\mathbf{i} + \frac{1}{4}(1-s)^{-1/2}\mathbf{j},$$
$$|\mathbf{T}'(s)| = \frac{1}{4}\sqrt{(1+s)^{-1} + (1-s)^{-1}} = \frac{1}{4}\sqrt{\frac{2}{1-s^2}},$$

  so, after some simplification,

$$\mathbf{N}(s) = \frac{1}{\sqrt{2}}(1-s)^{1/2}\mathbf{i} + \frac{1}{\sqrt{2}}(1+s)^{1/2}\mathbf{j}.$$

- (d) We compute that

$$\begin{aligned}\mathbf{B}(s) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{2}(1+s)^{1/2} & -\frac{1}{2}(1-s)^{1/2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}(1-s)^{1/2} & \frac{1}{\sqrt{2}}(1+s)^{1/2} & 0 \end{vmatrix} \\ &= -\frac{1}{2}(1+s)^{1/2}\mathbf{i} + \frac{1}{2}(1-s)^{1/2}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}\end{aligned}$$

- (e) Since $\mathbf{r}(s)$ is an arclength parametrization, we have

$$\kappa(s) = |\mathbf{T}'(s)| = \frac{1}{4}\sqrt{\frac{2}{1-s^2}}.$$

3. Let

$$\mathbf{F}(x,y,z) = y^2\mathbf{i} + (2xy+z^2)\mathbf{j} + (2yz+3z^2)\mathbf{k}.$$

(a) Compute $\nabla \times \mathbf{F}$.

(b) Find a function f such that $\mathbf{F} = \nabla f$.

(c) Let C be the oriented line with initial point $(1,-1,2)$ and terminal point $(-1,1,-1)$. Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

(d) True or False: "The line integral of $\mathbf{F}$ around any closed curve is zero."

Solution.

- (a) We have

$$\begin{aligned}\nabla \times \mathbf{F}(x,y,z) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & 2xy+z^2 & 2yz+3z^2 \end{vmatrix} \\ &= (2z-2z)\mathbf{i} - (0-0)\mathbf{j} + (2y-2y)\mathbf{k} \\ &= \mathbf{0}\end{aligned}$$

  Since $\nabla \times \mathbf{F} = \mathbf{0}$ on a simply connected region, the vector field $\mathbf{F}$ is conservative.

- (b) If $\mathbf{F} = \nabla f$, then by equating x-components and integrating, we get that

$$\frac{\partial f}{\partial x} = y^2, \qquad f(x,y,z) = xy^2 + g(y,z),$$

  where $g(y,z)$ is a function of integration. Equating y-components, we get

$$\frac{\partial f}{\partial y} = 2xy + z^2.$$

  Using the previous expression for f in this equation, we get that

$$2xy + \frac{\partial g}{\partial y} = 2xy + z^2,$$

  so, after simplifying and integrating, we get

$$\frac{\partial g}{\partial y} = z^2, \qquad g(y,z) = yz^2 + h(z),$$

where $h(z)$ is a function of integration. Finally, equating z-components

$$\frac{\partial f}{\partial z} = 2yz + 3z^2,$$

and using the previous expressions for

$$f(x, y, z) = xy^2 + yz^2 + h(z),$$

we get that

$$2yz + \frac{dh}{dz} = 2yz + 3z^2,$$

so

$$\frac{dh}{dz} = 3z^2, \qquad h(z) = z^3 + C,$$

where C is a constant of integration. It follows that

$$f(x, y, z) = xy^2 + yz^2 + z^3 + C.$$

- (c) Since $\mathbf{F} = \nabla f$ is conservative,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(-1, 1, -1) - f(1, -1, 2) = (-1) - 5 = -6.$$

- (d) True. A vector field in some region D is conservative if and only if its line integral around every closed curve in D is zero.

4. (a) Let the curve C be the triangle in the (x, y)-plane with vertices $(0, 0)$, $(1, 3)$, and $(0, 3)$, oriented counter-clockwise. Use Green's theorem to write the line integral

$$I = \oint_C \sqrt{x+y}\, dx$$

as an area integral over the interior R of C.

(b) Evaluate the area integral in (a). Compare your answer with your answer to Exercise 16, §16.2.

Solution.

- (a) An application of Green's theorem

$$\int\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \oint_C (M\, dx + N\, dy)$$

with $M = \sqrt{x+y}$ and $N = 0$, gives

$$\oint_C \sqrt{x+y}\, dx = -\int\int_R \frac{\partial}{\partial y}\sqrt{x+y}\, dxdy = -\frac{1}{2}\int\int_R \frac{1}{\sqrt{x+y}}\, dxdy.$$

- The triangle R can be described by $0 \le y \le 3$, $0 \le x \le y/3$, so

$$\begin{aligned}
-\frac{1}{2}\int\int_R \frac{1}{\sqrt{x+y}}\, dxdy &= -\frac{1}{2}\int_0^3 \left(\int_0^{y/3} \frac{1}{\sqrt{x+y}}\, dx\right) dy \\
&= -\frac{1}{2}\int_0^3 \left[2\sqrt{x+y}\right]_0^{y/3}\, dy \\
&= \left(1 - \frac{2}{\sqrt{3}}\right)\int_0^3 \sqrt{y}\, dy \\
&= 2\sqrt{3} - 4
\end{aligned}$$

- The area and line integrals agree. If C_1 is the line from $(0, 0)$ to $(1, 3)$, C_2 is the line from $(1, 3)$ to $(0, 3)$, and C_3 is the line from $(0, 3)$ to $(0, 0)$, one finds that

$$\int_{C_1} \sqrt{x+y}\, dx = \frac{4}{3}, \quad \int_{C_2} \sqrt{x+y}\, dx = 2\sqrt{3} - \frac{16}{3}, \quad \int_{C_3} \sqrt{x+y}\, dx = 0,$$

$$\oint_C \sqrt{x+y}\, dx = \frac{4}{3} + \left(2\sqrt{3} - \frac{16}{3}\right) + 0 = 2\sqrt{3} - 4.$$

5. Let R be the annulus $a^2 \leq x^2 + y^2 \leq b^2$ in the (x, y)-plane, where $0 < a < 1 < b$, and let $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ be the vector field on R with

$$M(x, y) = -\frac{y}{x^2 + y^2}, \qquad N(x, y) = \frac{x}{x^2 + y^2}.$$

(a) Show that

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \qquad \text{for all } (x, y) \text{ in } R.$$

(b) Let the curve C be circle $x^2 + y^2 = 1$ in R, oriented counter-clockwise. Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

Is $\mathbf{F}$ a conservative vector field on R?

(c) Explain why the results in (a) and (b) are consistent with Green's theorem.

Solution.

- (a) This equation follows by computing the derivatives and simplifying the result.
- (b) We parametrize the circle by $x = \cos t$, $y = \sin t$ with $0 \leq t \leq 2\pi$. Then $x^2 + y^2 = 1$, $dx = -(\sin t)dt$, $dy = (\cos t)dt$, and

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy \\ &= \int_0^{2\pi} \left(\sin^2 t + \cos^2 t\right) \, dt \\ &= 2\pi.\end{aligned}$$

- (c) Let C_a, C_1 be the circles with radii a and 1, oriented counter-clockwise, and let R_a be the annulus $a^2 \leq x^2 + y^2 \leq 1$. Then Green's theorem implies that

$$\int\int_{R_a} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \, dxdy = \oint_{C_1} (M\, dx + N\, dy) - \oint_{C_a} (M\, dx + N\, dy)$$

From (a), the left-hand side of this equation is zero, so the theorem implies that the circulation of $\mathbf{F}$ around C_1 is equal to the circulation around C_a,

$$\oint_{C_1} (M\,dx + N\,dy) = \oint_{C_a} (M\,dx + N\,dy)\,,$$

but it doesn't imply that the circulation is zero. We can't take $a = 0$ in this argument, since the vector field $\mathbf{F}$ isn't differentiable (or defined) at $(x, y) = (0, 0)$, and we can't apply Green's theorem to the entire circular region $x^2 + y^2 \leq 1$.