Solutions: Midterm 2
Math 21D, Fall 2019

1. [15pts] Suppose that C is the parametric curve with position vector

$$\mathbf{r}(t) = (e^t - t)\mathbf{i} + (4e^{t/2})\mathbf{j} - \mathbf{k}, \qquad 0 \le t \le 2.$$

Compute the arclength of C.

Solution.

- The arclength L of C is

$$L = \int_0^2 |\mathbf{r}'(t)|\, dt.$$

- We compute that

$$\begin{aligned} \mathbf{r}'(t) &= (e^t - 1)\mathbf{i} + (2e^{t/2})\mathbf{j} \\ |\mathbf{r}'(t)| &= \sqrt{(e^t - 1)^2 + 4e^t} = \sqrt{(e^t + 1)^2} = e^t + 1. \end{aligned}$$

- It follows that

$$\begin{aligned} L &= \int_0^2 (e^t + 1)\, dt \\ &= \left[e^t + t\right]_0^2 \\ &= e^2 + 1. \end{aligned}$$

2. [15pts] The position vector as a function of time t of a particle moving in space is given by $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$.s (a) Find the velocity $\mathbf{v}(t)$, speed $|\mathbf{v}(t)|$, unit tangent vector $\mathbf{T}(t)$, and acceleration $\mathbf{a}(t)$ of the particle.

(b) Find the curvature κ of the particle path at the point $(1, 1, 1)$. You don't need to compute the curvature at general points on the path, and you can use the formula

$$\kappa(t) = \frac{|\mathbf{r}'(\mathbf{t}) \times \mathbf{r}''(\mathbf{t})|}{|\mathbf{r}'(t)|^3}.$$

Solution.

- We have

$$\begin{aligned} \mathbf{v}(t) &= \mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}, \\ |\mathbf{v}(t)| &= \sqrt{1 + 4t^2 + 9t^4}, \\ \mathbf{T}(t) &= \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}}{\sqrt{1 + 4t^2 + 9t^4}}, \\ \mathbf{a}(t) &= \mathbf{r}''(t) = 2\mathbf{j} + 6t\mathbf{k}. \end{aligned}$$

- The point $(1, 1, 1)$ corresponds to $t = 1$. We have

$$\mathbf{r}'(1) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}, \qquad |\mathbf{r}'(1)| = \sqrt{14},$$
$$\mathbf{r}''(1) = 2\mathbf{j} + 6\mathbf{k}.$$

  It follows that

$$\mathbf{r}'(1) \times \mathbf{r}''(1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 0 & 2 & 6 \end{vmatrix} = 6\mathbf{i} - 6\mathbf{j} + 2\mathbf{k},$$

  so $|\mathbf{r}'(1) \times \mathbf{r}''(1)| = \sqrt{76}$, and

$$\kappa(1) = \frac{\sqrt{76}}{14^{3/2}} = \frac{1}{14}\sqrt{\frac{38}{7}}.$$

3. [15pts] (a) Say which one of the following three line integrals in the plane is path independent, and justify your answer:

$$(1) \int_C -ydx + xdy; \qquad (2) \int_C ydx + xdy; \qquad (3) \int_C (x+y)dx.$$

(b) Find a potential function for the line integral in (a) that is path independent and evaluate the line integral when C is the section of the parabola $y = x^2$ from the initial point $(1,1)$ to the terminal point $(2,4)$.

Solution.

- On a simply connected region (like the plane), the line integral

  $$\int_C M dx + N dy$$

  is path independent if and only if

  $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

  This condition is satisfied by (2) but not by (1) or (3), so (2) is the only path-independent line integral.

- We have $ydx + xdy = d(xy)$, so it is an exact differential form; equivalently the vector field

  $$y\mathbf{i} + x\mathbf{j} = \nabla(xy)$$

  is a conservative vector field with potential xy.

- It follows that

  $$\int_C ydx + xdy = [xy]_{(1,1)}^{(2,4)} = 7.$$

- Alternatively, with a little more work, we could evaluate the line integral by parameterizing C e.g., as $x = t$, $y = t^2$ with $1 \le t \le 2$, and evaluating the resulting t-integral:

  $$\int_C ydx + xdy = \int_1^2 3t^2\, dt = [t^3]_1^2 = 7.$$

4. [15pts] Let

$$\mathbf{F}(x,y,z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$$

and let C be the parametric curve $x = t$, $y = t^2$, $z = t^3$ with $0 \le t \le 1$. Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

Solution.

- We have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt.$$

- We compute that

$$\begin{aligned} \mathbf{F}(\mathbf{r}(t)) &= t^3\mathbf{i} + t^5\mathbf{j} + t^4\mathbf{k}, \\ \mathbf{r}'(t) &= \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k} \\ \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= t^3 + 5t^6, \end{aligned}$$

  so

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 (t^3 + 5t^6)\, dt \\ &= \left[\frac{1}{4}t^4 + \frac{5}{7}t^7\right]_0^1 \\ &= \frac{27}{28}. \end{aligned}$$

5. [15pts] Use Green's theorem to compute the outward flux $\int_C \mathbf{F} \cdot \mathbf{n}\, ds$ of

$$\mathbf{F}(x, y) = x^3\mathbf{i} + y^3\mathbf{j}$$

across the circle $x^2 + y^2 = 9$.

Solution.

- From the flux form of Green's theorem, if a simple closed curve C encloses the region R and $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$, then

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int\int_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy.$$

  For the given $\mathbf{F}$ and C, it follows that

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = 3 \int\int_R \left(x^2 + y^2\right) dxdy,$$

  where R is the disc $x^2 + y^2 \leq 9$.

- To evaluate this integral, we use polar coordinates with area element $dxdy = rdrd\theta$, which gives

$$\begin{aligned} \oint_C \mathbf{F} \cdot \mathbf{n}\, ds &= 3 \int_0^{2\pi} \int_0^3 r^2 \cdot rdrd\theta \\ &= 3 \cdot 2\pi \cdot \left[\frac{1}{4} r^4 \right]_0^3 \\ &= \frac{3^5}{2}\pi \\ &= \frac{243}{2}\pi. \end{aligned}$$

6. [15pts] Use Green's theorem to compute the circulation $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ of

$$\mathbf{F}(x,y) = (x^2 + y^2)\mathbf{i} - 2xy\mathbf{j}$$

around the triangle with vertices $(0,0)$, $(1,0)$, and $(0,2)$, oriented counter-clockwise.

Solution.

- From the circulation form of Green's theorem, if a simple closed curve C encloses the region R and $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$, then

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \int\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

  For the given $\mathbf{F}$ and C, it follows that

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \int\int_R (-2y - 2y)\ dxdy = -4 \int\int_R y\, dxdy$$

  where R is the triangle enclosed by C.

- The triangle R is described by $0 \le x \le 1$, $0 \le y \le 2 - 2x$, so

$$\begin{aligned} \int\int_R y\, dxdy &= \int_0^1 \int_0^{2-2x} y\, dydx \\ &= \int_0^1 \left[\frac{1}{2} y^2 \right]_0^{2-2x} dx \\ &= 2 \int_0^1 (1-x)^2\, dx \\ &= 2 \left[-\frac{1}{3}(1-x)^3 \right]_0^1 \\ &= \frac{2}{3}, \end{aligned}$$

  and

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = -\frac{8}{3}.$$