Calculus

Math 21D, Fall 2019

Sample Questions: Midterm II

1. (a) Write down a parametric equation for the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a, b > 0$ are constants.

(b) Use the parametrization in (a) to obtain an integral for the arclength of the ellipse. Do not attempt to evaluate the integral.

2. Let C be a curve parametrized by $-1 < s < 1$ with position vector

$$\mathbf{r}(s) = \frac{1}{3}(1+s)^{3/2}\mathbf{i} + \frac{1}{3}(1-s)^{3/2}\mathbf{j} + \frac{1}{\sqrt{2}}s\mathbf{k}.$$

(a) Show that s is an arclength parameter for the curve.

(b) Find the unit tangent vector $\mathbf{T}$.

(c) Find the unit normal vector $\mathbf{N}$.

(d) Find the binormal vector $\mathbf{B} = \mathbf{T} \times \mathbf{N}$.

(e) Find the curvature κ.

3. Let

$$\mathbf{F}(x, y, z) = y^2\mathbf{i} + (2xy + z^2)\mathbf{j} + (2yz + 3z^2)\mathbf{k}.$$

(a) Compute $\nabla \times \mathbf{F}$.

(b) Find a function f such that $\mathbf{F} = \nabla f$.

(c) Let C be the oriented line with initial point $(1, -1, 2)$ and terminal point $(-1, 1, -1)$. Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

(d) True or False: "The line integral of $\mathbf{F}$ around any closed curve is zero."

4. (a) Let the curve C be the triangle in the (x, y)-plane with vertices $(0, 0)$, $(1, 3)$, and $(0, 3)$, oriented counter-clockwise. Use Green's theorem to write the line integral

$$I = \oint_C \sqrt{x + y}\, dx$$

as an area integral over the interior R of C.

(b) Evaluate the area integral in (a). Compare your answer with your answer to Exercise 16, §16.2.

5. Let R be the annulus $a^2 \leq x^2 + y^2 \leq b^2$ in the (x, y)-plane, where $0 < a < 1 < b$, and let $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ be the vector field on R with

$$M(x, y) = -\frac{y}{x^2 + y^2}, \qquad N(x, y) = \frac{x}{x^2 + y^2}.$$

(a) Show that

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \qquad \text{for all } (x, y) \text{ in } R.$$

(b) Let the curve C be circle $x^2 + y^2 = 1$ in R, oriented counter-clockwise. Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

Is $\mathbf{F}$ a conservative vector field on R?

(c) Explain why the results in (a) and (b) are consistent with Green's theorem.