1. [15pts] (a) Let $R$ be the triangle in the $(x,y)$-plane with vertices $(0,0)$, $(1,3)$, $(0,3)$. Sketch $R$.
(b) Write the double integral
\[
\int \int_{R} (3x + 2y^2) \, dA
\]
as an iterated integral in the order $dxdy$ (integrate with respect to $x$ first followed by $y$). DO NOT EVALUATE.
(c) Write the double integral in (b) as an iterated integral in the order $dydx$ (integrate with respect to $y$ first followed by $x$). DO NOT EVALUATE.

Solution.

• (a) The triangle $R$ is shown in the figure.

• (b) The triangle $R$ is described by $0 \leq y \leq 3$, $0 \leq x \leq y/3$ so

\[
\int \int_{R} (3x + 2y^2) \, dA = \int_{0}^{3} \int_{0}^{y/3} (3x + 2y^2) \, dx \, dy.
\]

• (b) The triangle $R$ is described by $0 \leq x \leq 1$, $3x \leq y \leq 3$ so

\[
\int \int_{R} (3x + 2y^2) \, dA = \int_{0}^{1} \int_{3x}^{3} (3x + 2y^2) \, dy \, dx.
\]
2. [15pts] (a) Evaluate the iterated integral
\[
\int_0^3 \int_{y^2-2y}^{4y-y^2} x \, dx \, dy.
\]
(b) Sketch the region of integration in the \((x, y)\)-plane for the integral in (a).

Solution.

• (a) We have
\[
\int_{y^2-2y}^{4y-y^2} x \, dx = \left[ \frac{1}{2} x^2 \right]_{y^2-2y}^{4y-y^2} = \frac{1}{2} \left[ (4y - y^2)^2 - (y^2 - 2y)^2 \right] = 6y^2 - 2y^3,
\]
\[
\int_0^3 \left( 6y^2 - 2y^3 \right) \, dy = \left[ 2y^3 - \frac{1}{2} y^4 \right]_0^3 = \frac{27}{2}.
\]

• (b) The region \( R \) is shown in the figure.
3. [15pts] Let $R$ be the region in $x \geq 0,$ $y \geq 0$ between the lines $y = 0,$ $y = x$ and the circles $x^2 + y^2 = 1,$ $x^2 + y^2 = 4$ (see figure). Use polar coordinates to evaluate the double integral

$$\int \int_R \frac{1}{(x^2 + y^2)^{3/2}} \, dA.$$ 

Solution.

- The region $R$ is described in polar coordinates by 

  $$0 \leq \theta \leq \pi/4, \quad 1 \leq r \leq 2.$$ 

- Since $x^2 + y^2 = r^2$ and $dA = r \, dr \, d\theta,$ we get 

  $$\int \int_R \frac{1}{(x^2 + y^2)^{3/2}} \, dA = \int_0^{\pi/4} \int_1^2 \frac{1}{r^3} \, r \, dr \, d\theta$$ 

  $$= \left( \int_0^{\pi/4} d\theta \right) \left( \int_1^2 \frac{1}{r^2} \, dr \right)$$ 

  $$= \frac{\pi}{4} \left[ -\frac{1}{r} \right]_1^2$$ 

  $$= \frac{\pi}{4} \left( -1 \right)^2$$ 

  $$= \frac{\pi}{8}.$$ 


Let $a, b, c > 0$ be positive constants and let $D$ be the rectangular box $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$.

(a) Evaluate the triple integral

$$\int \int \int_D (x + y + z) \, dV.$$ 

(b) What is the average value of $x + y + z$ over $D$?

Solution.

• (a) We have (any other order of integration works as well)

$$\int \int \int_D (x + y + z) \, dV = \int_0^a \int_0^b \int_0^c (x + y + z) \, dz \, dy \, dx.$$ 

Then

$$\int_0^c (x + y + z) \, dz = \left[ xz + yz + \frac{1}{2} z^2 \right]_0^c = xc + yc + \frac{1}{2} c^2,$$

$$\int_0^b \left( xc + yc + \frac{1}{2} c^2 \right) \, dy = \left[ xyc + \frac{1}{2} y^2 c + \frac{1}{2} yc^2 \right]_0^b = xbc + \frac{1}{2} b^2 c + \frac{1}{2} bc^2,$$

$$\int_0^a \left( xbc + \frac{1}{2} b^2 c + \frac{1}{2} bc^2 \right) \, dx = \left[ \frac{1}{2} x^2 bc + \frac{1}{2} x b^2 c + \frac{1}{2} x bc^2 \right]_0^a = \frac{1}{2} a^2 bc + \frac{1}{2} ab^2 c + \frac{1}{2} abc^2,$$

$$\int \int \int_D (x + y + z) \, dV = \frac{1}{2} abc (a + b + c).$$

• (b) The volume of $D$ is $abc$, so the average value of $x + y + z$ on $D$ is

$$\frac{1}{\text{Volume}(D)} \int \int \int_D (x + y + z) \, dV = \frac{1}{2} (a + b + c).$$
5. [20pts] Let \( D \) be the upper-half of the unit sphere \( x^2 + y^2 + z^2 \leq 1, \ z \geq 0, \) and let \( I \) be the triple integral

\[
I = \iiint_D z(x^2 + y^2) \, dV.
\]

(a) Write \( I \) as an iterated integral in cylindrical coordinates in the order \( dzd\theta dr \) (integrate with respect to \( z \) first). DO NOT EVALUATE.
(b) Write \( I \) as an iterated integral in spherical coordinates in the order \( d\theta d\phi d\rho \). DO NOT EVALUATE.

Solution.

• (a) Since \( x^2 + y^2 + z^2 = 1 \) gives \( z^2 = 1 - r^2 \), the hemisphere is described in cylindrical coordinates by

\[
0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq \sqrt{1 - r^2}.
\]

Using \( x^2 + y^2 = r^2 \) and the cylindrical volume element

\[
dV = r \, dzd\theta dr,
\]

we get that

\[
I = \int_0^1 \int_0^{2\pi} \int_0^{\sqrt{1-r^2}} zr^2 \cdot r \, dzd\theta dr.
\]

• (b) The hemisphere is described in spherical coordinates by

\[
0 \leq \rho \leq 1, \quad 0 \leq \phi \leq \frac{\pi}{2}, \quad 0 \leq \theta \leq 2\pi.
\]

Using \( z = \rho \cos \phi, \ x^2 + y^2 = \rho^2 \sin^2 \phi \), and the spherical volume element

\[
dV = \rho^2 \sin \phi \, d\theta d\phi d\rho,
\]

we get that

\[
I = \int_0^1 \int_0^{\pi/2} \int_0^{2\pi} \rho \cos \phi \cdot \rho^2 \sin^2 \phi \cdot \rho^2 \sin \phi \, d\theta d\phi d\rho.
\]
6. [10pts] Let $R$ be closed, bounded region in the $(x,y)$-plane and $f(x,y)$ an integrable function on $R$. Explain how $\iint_{R} f \, dA$ is defined in terms of Riemann sums.

Solution.

- For every partition $\mathcal{P}$ of $R$ into rectangles $R_1, R_2, \ldots, R_n$, choose any point $(x_j, y_j)$ in each rectangle $R_j$ of the partition. Then

$$\iint_{R} f \, dA = \lim_{\|\mathcal{P}\| \to 0} \sum_{j=1}^{n} f(x_j, y_j) \Delta A_j,$$

where $\|\mathcal{P}\|$ is the maximum side-length of the rectangles $R_j$ in the partition $\mathcal{P}$ and $\Delta A_j$ is the area of $R_j$.

Optional Remark. A precise way to state this definition is as follows. A function $f(x,y)$ is integrable on $R$ with

$$\iint_{R} f \, dA = L$$

if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left| \sum_{j=1}^{n} f(x_j, y_j) \Delta A_j - L \right| < \epsilon$$

for every partition $\mathcal{P}$ of $R$ with $\|\mathcal{P}\| < \delta$ and every choice of points $(x_j, y_j)$ in $R_j$. 

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