Midterm 1: Solutions
Math 21D, Fall 2019

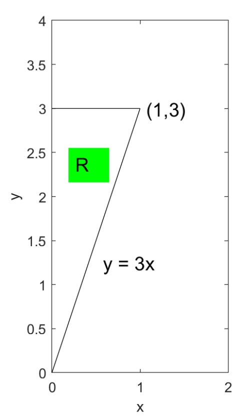
1. [15pts] (a) Let R be the triangle in the (x, y)-plane with vertices $(0, 0)$, $(1, 3)$, $(0, 3)$. Sketch R.
(b) Write the double integral

$$\int\int_R \left(3x + 2y^2\right)\, dA$$

as an iterated integral in the order $dxdy$ (integrate with respect to x first followed by y). DO NOT EVALUATE.
(c) Write the double integral in (b) as an iterated integral in the order $dydx$ (integrate with respect to y first followed by x). DO NOT EVALUATE.

Solution.

- (a) The triangle R is shown in the figure.



- (b) The triangle R is described by $0 \le y \le 3$, $0 \le x \le y/3$ so

$$\int\int_R \left(3x + 2y^2\right)\, dA = \int_0^3 \int_0^{y/3} \left(3x + 2y^2\right)\, dxdy.$$

- (b) The triangle R is described by $0 \le x \le 1$, $3x \le y \le 3$ so

$$\int\int_R \left(3x + 2y^2\right)\, dA = \int_0^1 \int_{3x}^3 \left(3x + 2y^2\right)\, dydx.$$

2. [15pts] (a) Evaluate the iterated integral

$$\int_0^3 \int_{y^2-2y}^{4y-y^2} x\, dxdy.$$

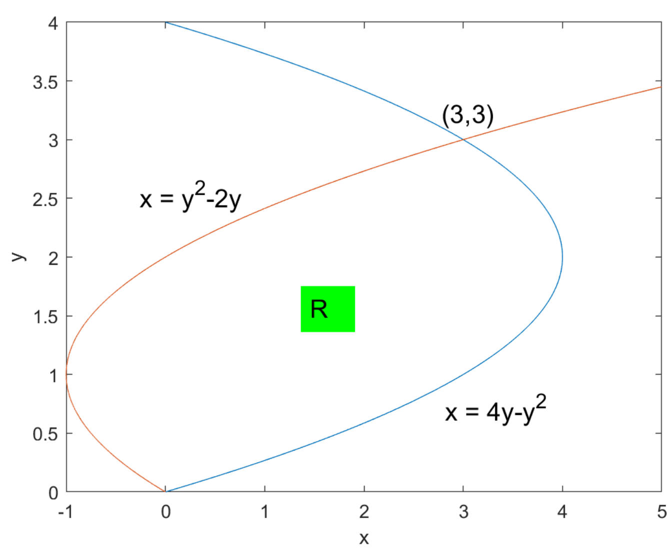
(b) Sketch the region of integration in the (x, y)-plane for the integral in (a).

Solution.

- (a) We have

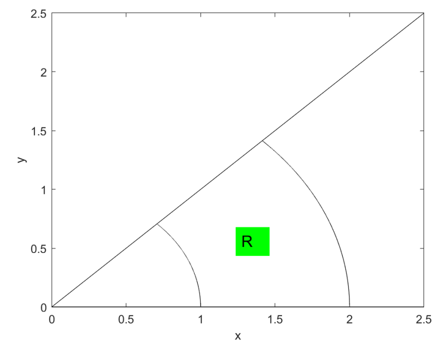
$$\begin{aligned} \int_{y^2-2y}^{4y-y^2} x\, dx &= \left[\frac{1}{2}x^2\right]_{y^2-2y}^{4y-y^2} \\ &= \frac{1}{2}\left[\left(4y-y^2\right)^2 - \left(y^2-2y\right)^2\right] \\ &= 6y^2 - 2y^3, \\ \int_0^3 \int_{y^2-2y}^{4y-y^2} x\, dxdy &= \int_0^3 \left(6y^2 - 2y^3\right)\, dy \\ &= \left[2y^3 - \frac{1}{2}y^4\right]_0^3 = \frac{27}{2}. \end{aligned}$$

- (b) The region R is shown in the figure.



3. [15pts] Let R be the region in $x \geq 0$, $y \geq 0$ between the lines $y = 0$, $y = x$ and the circles $x^2 + y^2 = 1$, $x^2 + y^2 = 4$ (see figure). Use polar coordinates to evaluate the double integral

$$\int\int_R \frac{1}{(x^2+y^2)^{3/2}}\, dA.$$



Solution.

- The region R is described in polar coordinates by

$$0 \leq \theta \leq \pi/4, \qquad 1 \leq r \leq 2.$$

- Since $x^2 + y^2 = r^2$ and $dA = r dr d\theta$, we get

$$\begin{aligned}
\int\int_R \frac{1}{(x^2+y^2)^{3/2}}\, dA &= \int_0^{\pi/4} \int_1^2 \frac{1}{r^3}\, r dr d\theta \\
&= \left(\int_0^{\pi/4} d\theta \right) \left(\int_1^2 \frac{1}{r^2}\, dr \right) \\
&= \frac{\pi}{4} \left[-\frac{1}{r} \right]_1^2 \\
&= \frac{\pi}{8}.
\end{aligned}$$

4. [15pts] Let $a, b, c > 0$ be positive constants and let D be the rectangular box $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$.

(a) Evaluate the triple integral

$$\int\int\int_D (x+y+z)\ dV.$$

(b) What is the average value of $x+y+z$ over D?

Solution.

- (a) We have (any other order of integration works as well)

$$\int\int\int_D (x+y+z)\ dV = \int_0^a \int_0^b \int_0^c (x+y+z)\ dzdydx.$$

  Then

$$\begin{aligned}
\int_0^c (x+y+z)\ dz &= \left[xz + yz + \frac{1}{2}z^2\right]_0^c \\
&= xc + yc + \frac{1}{2}c^2, \\
\int_0^b \left(xc + yc + \frac{1}{2}c^2\right)\ dy &= \left[xyc + \frac{1}{2}y^2c + \frac{1}{2}yc^2\right]_0^b \\
&= xbc + \frac{1}{2}b^2c + \frac{1}{2}bc^2 \\
\int_0^a \left(xbc + \frac{1}{2}b^2c + \frac{1}{2}bc^2\right)\ dx &= \left[\frac{1}{2}x^2bc + \frac{1}{2}xb^2c + \frac{1}{2}xbc^2\right]_0^a \\
&= \frac{1}{2}a^2bc + \frac{1}{2}ab^2c + \frac{1}{2}abc^2, \\
\int\int\int_D (x+y+z)\ dV &= \frac{1}{2}abc\,(a+b+c)\,.
\end{aligned}$$

- (b) The volume of D is abc, so the average value of $x+y+z$ on D is

$$\frac{1}{\text{Volume}(D)}\int\int\int_D (x+y+z)\ dV = \frac{1}{2}\,(a+b+c)\,.$$

5. [20pts] Let D be the upper-half of the unit sphere $x^2+y^2+z^2 \leq 1$, $z \geq 0$, and let I be the triple integral

$$I = \int\int\int_D z\left(x^2+y^2\right)\, dV.$$

(a) Write I as an iterated integral in cylindrical coordinates in the order $dzd\theta dr$ (integrate with respect to z first). DO NOT EVALUATE.

(b) Write I as an iterated integral in spherical coordinates in the order $d\theta d\phi d\rho$. DO NOT EVALUATE.

Solution.

- (a) Since $x^2+y^2+z^2 = 1$ gives $z^2 = 1-r^2$, the hemisphere is described in cylindrical coordinates by

$$0 \leq r \leq 1, \qquad 0 \leq \theta \leq 2\pi, \qquad 0 \leq z \leq \sqrt{1-r^2}.$$

  Using $x^2+y^2 = r^2$ and the cylindrical volume element

$$dV = r\, dzd\theta dr,$$

  we get that

$$I = \int_0^1 \int_0^{2\pi} \int_0^{\sqrt{1-r^2}} zr^2 \cdot r\, dzd\theta dr.$$

- (b) The hemisphere is described in spherical coordinates by

$$0 \leq \rho \leq 1, \qquad 0 \leq \phi \leq \frac{\pi}{2}, \qquad 0 \leq \theta \leq 2\pi.$$

  Using $z = \rho\cos\phi$, $x^2+y^2 = \rho^2\sin^2\phi$, and the spherical volume element

$$dV = \rho^2 \sin\phi\, d\theta d\phi d\rho,$$

  we get that

$$I = \int_0^1 \int_0^{\pi/2} \int_0^{2\pi} \rho\cos\phi \cdot \rho^2\sin^2\phi \cdot \rho^2\sin\phi\, d\theta d\phi d\rho.$$

6. [10pts] Let R be closed, bounded region in the (x, y)-plane and $f(x, y)$ an integrable function on R. Explain how $\int\int_R f\,dA$ is defined in terms of Riemann sums.

Solution.

- For every partition $\mathcal{P}$ of R into rectangles $R_1, R_2, \ldots, R_n$, choose any point (x_j, y_j) in each rectangle R_j of the partition. Then

$$\int\int_R f dA = \lim_{\|\mathcal{P}\|\to 0} \sum_{j=1}^{n} f(x_j, y_j)\Delta A_j,$$

  where $\|\mathcal{P}\|$ is the maximum side-length of the rectangles R_j in the partition $\mathcal{P}$ and ΔA_j is the area of R_j.

Optional Remark. A precise way to state this definition is as follows. A function $f(x, y)$ is integrable on R with

$$\int\int_R f dA = L$$

if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left| \sum_{j=1}^{n} f(x_j, y_j)\Delta A_j - L \right| < \epsilon$$

for every partition $\mathcal{P}$ of R with $\|\mathcal{P}\| < \delta$ and every choice of points (x_j, y_j) in R_j.