Calculus
Math 21D, Fall, 2019
Solutions: Sample Questions, Midterm I

1. (a) Evaluate the iterated integral

$$\int_0^1 \int_{x^3}^{\sqrt{x}} y \, dy dx.$$

(b) Sketch the region of integration R in the (x, y)-plane determined by the iterated integral in (a) and rewrite the integral with the order of integration reversed. Evaluate it, and show that you get the same result as in (a).

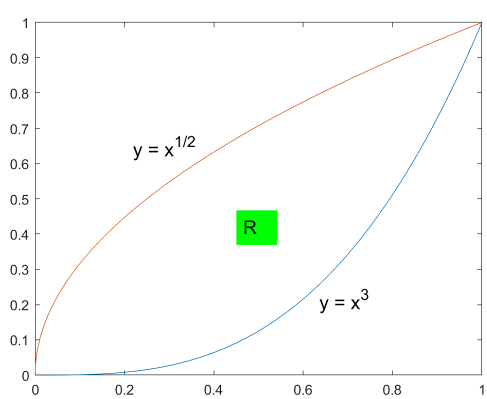
Solution.

- (a) We have

$$\begin{aligned}\int_0^1 \int_{x^3}^{\sqrt{x}} y \, dy dx &= \int_0^1 \left[\frac{1}{2} y^2\right]_{x^3}^{\sqrt{x}} dx = \frac{1}{2}\int_0^1 \left(x - x^6\right) dx \\ &= \frac{1}{2}\left[\frac{1}{2}x^2 - \frac{1}{7}x^7\right]_0^1 = \frac{5}{28}\end{aligned}$$

- (b) The region is given by $0 \le y \le 1$, $y^2 \le x \le y^{1/3}$ so

$$\begin{aligned}\int_0^1 \int_{x^3}^{\sqrt{x}} y \, dy dx &= \int_0^1 \int_{y^2}^{y^{1/3}} y \, dx dy = \int_0^1 \left[xy\right]_{y^2}^{y^{1/3}} dy \\ &= \int_0^1 \left(y^{4/3} - y^3\right) dx = \left[\frac{3}{7}y^{7/3} - \frac{1}{4}y^4\right]_0^1 = \frac{5}{28}\end{aligned}$$



2. (a) Let $0 < a < b$ and $c > 0$ be positive constants. Find the volume under the surface $xyz = c$ over the square $a \le x \le b$, $a \le y \le b$.

(b) How does the volume over the square $a \le x \le b$, $a \le y \le b$ compare with the volume over the square $2a \le x \le 2b$, $2a \le y \le 2b$? Explain.

Solution.

- (a) The volume V is given by

$$\begin{aligned} V &= \int_a^b \int_a^b z \, dxdy = \int_a^b \int_a^b \frac{c}{xy} \, dxdy \\ &= c \left(\int_a^b \frac{dx}{x} \right) \left(\int_a^b \frac{dy}{y} \right) \\ &= c \, (\ln b - \ln a)^2 \\ &= c \left(\ln \frac{b}{a} \right)^2 \end{aligned}$$

- (b) The volumes are the same. This is because the area of the base increases by a factor of 4 when we double a and b but the average height of the volume decreases by a factor of 4.

- Note that c has dimensions of length cubed and b/a is dimensionless (since it's a ratio of two lengths), so the solution for V has the correct dimensions. Also note that any quantity appearing inside a logarithm must be dimensionless, otherwise a change in units would rescale the quantity, leading to a different answer.

3. (a) Let R be the circle $x^2 + y^2 \leq \pi$ of radius $\sqrt{\pi}$. Evaluate

$$\int\int_R \sin\left(x^2 + y^2\right)\, dA.$$

(b) What is the average value of $\sin\left(x^2 + y^2\right)$ over R?

Solution.

- (a) We use polar coordinates (r, θ). Then R is the region $0 \leq r \leq \sqrt{\pi}$, $0 \leq \theta \leq 2\pi$, and

$$\int\int_R \sin\left(x^2 + y^2\right)\, dA = \int_0^{2\pi} \int_0^{\sqrt{\pi}} \sin(r^2)\, r dr d\theta.$$

  Using the substitution $u = r^2$, with $du = 2r d\, r$, we get that

$$\int_0^{\sqrt{\pi}} \sin(r^2)\, r dr = \frac{1}{2} \int_0^{\pi} \sin u\, du = \frac{1}{2} \left[-\cos u\right]_0^{\pi} = 1,$$
$$\int\int_R \sin\left(x^2 + y^2\right)\, dA = \int_0^{2\pi} d\theta = 2\pi.$$

- (b) The area of a circle with radius $\sqrt{\pi}$ is $\pi(\sqrt{\pi})^2 = \pi^2$, so the average value of $\sin\left(x^2 + y^2\right)$ over R is

$$\text{average} = \frac{1}{\pi^2} \int\int_R \sin\left(x^2 + y^2\right)\, dA = \frac{2}{\pi}.$$

4. A wedge of cheese is cut from a cylinder that is 4cm high and 6cm in radius. If the angle of the wedge at the center of the cylinder is $\pi/6$ and the density of the cheese is $2\text{gm}/\text{cm}^3$, compute the mass of the wedge of cheese (include units).

Solution.

- We use cylindrical coordinates (r, θ, z). The wedge is described by

$$0 \le r \le 6, \qquad 0 \le \theta \le \frac{\pi}{6}, \qquad 0 \le z \le 4,$$

  so the mass M of the cheese is

$$\begin{aligned} M &= \int_0^4 \int_0^{\pi/6} \int_0^6 2\, r dr d\theta dz \\ &= \left(\int_0^6 2\, r dr \right) \left(\int_0^{\pi/6} d\theta \right) \left(\int_0^4 dz \right) \\ &= 6^2 \cdot \frac{\pi}{6} \cdot 4 \\ &= 24\pi \text{ gm}. \end{aligned}$$

5. Transform the following iterated integral to spherical coordinates and evaluate it:

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\,dzdydx.$$

Solution.

- The integration region

$$0 \le z \le \sqrt{9-x^2-y^2}, \quad -\sqrt{9-x^2} \le y \le \sqrt{9-x^2}, \quad -3 \le x \le 3$$

corresponds to the upper hemisphere of the sphere $x^2+y^2+z^2 \le 9$ of radius 3 (with $z \ge 0$ and x, y in the circle $x^2+y^2 \le 9$ with any sign).

- In spherical coordinates, this region is described by

$$0 \le \rho \le 3, \qquad 0 \le \phi \le \frac{\pi}{2}, \qquad 0 \le \theta \le 2\pi.$$

Using

$$z = \rho\cos\phi, \qquad \sqrt{x^2+y^2+z^2} = \rho, \qquad dxdydz = \rho^2 \sin\phi\, d\rho d\phi d\theta,$$

we get that

$$\begin{aligned}
&\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\,dzdydx \\
&\quad = \int_0^{2\pi}\int_0^{\pi/2}\int_0^3 \rho\cos\phi\cdot\rho\cdot\rho^2\sin\phi\,d\rho d\phi d\theta \\
&\quad = \left(\int_0^3 \rho^4\,d\rho\right)\left(\int_0^{\pi/2}\cos\phi\sin\phi\,d\phi\right)\left(\int_0^{2\pi} d\theta\right) \\
&\quad = \frac{3^5}{5}\cdot\frac{1}{2}\cdot 2\pi \\
&\quad = \frac{243\pi}{5}
\end{aligned}$$

6. Suppose that a function $f(x, y)$ is defined on a closed bounded region R in the plane.

(a) Let $\mathcal{P}$ be a partition of R into rectangles $R_1, R_2, \ldots, R_n$. Define the Riemann sums $S(f, \mathcal{P})$ of f with respect to $\mathcal{P}$.

(b) Define what it means for f to be integrable on R and define $\int\int_R f dA$.

(c) If f is integrable on R and c is a constant, use the definition in (b) to show that

$$\int\int_R cf dA = c\int\int_R f dA.$$

Solution.

- (a) Choose a point (x_j, y_j) in each rectangle R_j of the partition. Then

$$S(f, \mathcal{P}) = \sum_{j=1}^{n} f(x_j, y_j)\Delta A_j$$

where ΔA_j is the area of R_j.

- (b) We define

$$\int\int_R f dA = \lim_{\|\mathcal{P}\|\to 0} \sum_{j=1}^{n} f(x_j, y_j)\Delta A_j$$

where $\|\mathcal{P}\|$ is the maximum side-length of the rectangles R_j in the partition $\mathcal{P}$. The function f is integrable on R is this limit exists however we choose the partitions $\mathcal{P}$ and the points (x_j, y_j) in the rectangles R_j.

- (c) We have

$$S(cf, \mathcal{P}) = \sum_{j=1}^{n} cf(x_j, y_j)\Delta A_j = c\sum_{j=1}^{n} f(x_j, y_j)\Delta A_j = cS(f, \mathcal{P}).$$

Using the fact that $\lim cS = c \lim S$ for any constant c, we get that

$$\begin{aligned}\int\int_R cf\, dA &= \lim_{\|\mathcal{P}\|\to 0} S(cf, \mathcal{P}) = \lim_{\|\mathcal{P}\|\to 0} cS(f, \mathcal{P}) = c\lim_{\|\mathcal{P}\|\to 0} S(f, \mathcal{P}) \\ &= c\int\int_R f\, dA.\end{aligned}$$