1. [10%] Say if the following ODEs are linear or nonlinear. By use of the theorems given in class, what can you say about the existence and uniqueness of a solution $y(t)$ of the ODEs with the initial condition $y(1) = -2$?

(a) $(t + 1)y' + (\cos t)y = e^t$.
(b) $(y + 1)y' + (\cos t)y = e^t$.

Solution.

• (a) Linear $[y' + p(t)y = g(t)]$. The coefficient functions

$$p(t) = \frac{\cos t}{t + 1}, \quad g(t) = \frac{e^t}{t + 1}$$

are continuous in the interval $-1 < t < \infty$, containing the initial time $t_0 = 1$, so a unique solution exists in $-1 < t < \infty$.

• (b) Nonlinear $[y' = f(t, y)]$. The function

$$f(t, y) = \frac{-\cos t y + e^t}{y + 1}$$

is continuous and has a continuous derivative with respect to $y$ except when $y = -1$. Since the initial value of $y$ is $-2$, a unique solution exists in some time interval containing $t_0 = 1$. 

2. [10%] Suppose that for certain continuous function \( p(t) \), \( g(t) \) and initial time \( t_0 \), the functions \( y_1(t) \), \( y_2(t) \) are solutions of the initial value problems

\[
\begin{align*}
y_1' + p(t)y_1 &= g(t), \quad y_1(t_0) = 0, \\
y_2' + p(t)y_2 &= 0, \quad y_2(t_0) = 1.
\end{align*}
\]

If \( c_1, c_2 \) are constants, show that the function

\[ y(t) = c_1y_1(t) + c_2y_2(t) \]

is the solution of the initial value problem

\[ y' + p(t)y = c_1 g(t), \quad y(t_0) = c_2. \]

**Solution.**

- Using the definition of \( y(t) \), the linearity of the derivative, and the ODEs satisfied by \( y_1(t), y_2(t) \), we get

\[
\begin{align*}
y' + p(t)y &= (c_1y_1 + c_2y_2)' + p(t) (c_1y_1 + c_2y_2) \\
&= c_1y_1' + c_2y_2' + c_1p(t)y_1 + c_2p(t)y_2 \\
&= c_1 (y_1' + p(t)y_1) + c_2 (y_2' + p(t)y_2) \\
&= c_1 \cdot g(t) + c_2 \cdot 0 \\
&= c_1g(t).
\end{align*}
\]

- Similarly, using the definition of \( y(t) \) and the initial conditions satisfied by \( y_1(t), y_2(t) \), we get

\[
\begin{align*}
y(t_0) &= c_1y_1(t_0) + c_2y_2(t_0) \\
&= c_1 \cdot 0 + c_2 \cdot 1 \\
&= c_2.
\end{align*}
\]
3. [20%] Suppose that $a$, $b$ are constants. Find the general solution of

$$y' + ay = b.$$ 

If $a > 0$, how does the solution behave as $t \to +\infty$?

Solution.

- Multiplying the ODE by the integrating factor $\mu(t) = e^{at}$ and rewriting the result, we get

$$(e^{at}y)' = be^{at}.$$ 

Integrating this equation and solving for $y(t)$, we get

$$y(t) = \frac{b}{a} + ce^{-at} \quad \text{if } a \neq 0,$$

and

$$y(t) = bt + c \quad \text{if } a = 0,$$

where $c$ is a constant of integration.

- If $a > 0$, then $e^{-at} \to 0$ as $t \to +\infty$, so

$$y(t) \to \frac{b}{a} \quad \text{as } t \to +\infty.$$ 

Remark. This equation is the general form of an autonomous linear first-order scalar ODE. If $a \neq 0$, it has a single equilibrium, $y = b/a$. This equilibrium is asymptotically stable if $a > 0$ and unstable if $a < 0$. Note that, unlike autonomous nonlinear ODEs, an autonomous linear ODE cannot have finitely many different equilibria. (If $a = 0$ and $b \neq 0$, the ODE has no equilibria, and if $a = b = 0$ every point $y = c$ is an equilibrium.)
4. [20%] (a) Find the solution of the initial value problem

\[ ty' + 3y = \frac{1}{t}, \quad t > 0, \]

\[ y(1) = y_0. \]

(b) For what initial values \( y_0 \) is the solution \( y(t) \) equal to zero for some \( t > 0 \)?

Solution.

• (a) The ODE is linear and first-order, so we can solve it by the integrating factor method. The standard form is

\[ y' + 3\frac{t}{t}y = \frac{1}{t^2}. \]

Multiplying the standard form of the ODE by the integrating factor

\[ \mu(t) = \exp \left( \int \frac{3}{t} dt \right) = \exp \left( 3 \ln t \right) = t^3, \]

and rewriting the left-hand side of the result as an exact derivative, we get

\[ \left( t^3 y \right)' = t. \]

Integrating this equation and solving for \( y \), we get

\[ y(t) = \frac{1}{2t} + \frac{C}{t^3}, \]

where \( C \) is a constant of integration.

• Imposing the initial condition \( y = y_0 \) when \( t = 1 \), and solving for \( C \), we find that

\[ C = y_0 - \frac{1}{2}. \]

Thus, the solution is

\[ y(t) = \frac{1}{2t} + \frac{y_0 - 1/2}{t^3}. \]

• (b) We have

\[ y(t) = \frac{1}{t} \left( \frac{1}{2} + \frac{y_0 - 1/2}{t^2} \right). \]

This vanishes at some \( t > 0 \) if and only if \( y_0 - 1/2 < 0 \) or

\[ y_0 < \frac{1}{2}. \]
5. [20%] (a) Solve the initial value problem

\[ ty' + y^2 = 0, \]
\[ y(1) = 1. \]

(b) What is the largest \( t \)-interval in which the solution exists?

**Solution.**

- (a) The equation is nonlinear and separable. Separating variables, we get

\[ -\int \frac{dy}{y^2} = \int \frac{dt}{t}. \]

Integration of this equation gives

\[ \frac{1}{y} = \ln t + C. \]

(Here, we write the integral \( \ln |t| \) as \( \ln t \), since the initial time \( t_0 = 1 \) is positive.)

- The initial condition, \( y = 1 \) at \( t = 1 \), implies that \( C = 1 \). Solving for \( y \), we find that the solution is

\[ y(t) = \frac{1}{\ln t + 1}. \]

- (b) The denominator vanishes when \( \ln t = -1 \) or \( t = 1/e > 0 \). The solution is therefore defined in the time interval

\[ \frac{1}{e} < t < +\infty. \]
6. [20%] Consider the ODE

\[ y' = y^2 - y^4. \]

(a) Find all equilibrium solutions.
(b) Sketch the phase line.
(c) Determine the stability of the equilibria you found in (a).

Solution.

• (a) The equilibria satisfy

\[ y^2 - y^4 = y^2 (1 - y^2) = 0, \]

so \( y = 0 \) or \( y = \pm 1 \).

• (b) The phase line looks like this:

\[ \begin{array}{ccccccccc}
\leftarrow & \cdot & \rightarrow & \cdot & \rightarrow & \cdot & \leftarrow \\
0
\end{array} \]

• (c) \( y = 0 \) is semistable, \( y = 1 \) is asymptotically stable, and \( y = -1 \) is unstable.