Ordinary Differential Equations
Math 22B-002, Fall 2014
Sample Final Exam

NAME....................................................................
SIGNATURE..........................................................
I.D. NUMBER.......................................................

No books, notes, or calculators. Show all your work

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1. [20 pts.] (a) Solve the following initial value problem for $y(t)$ in $t > 0$:

$$y' - \frac{2}{t} y = t, \quad y(1) = 2.$$ 

(b) How does $y(t)$ behave as $t \to 0^+$?
2. [20 pts.] (a) Solve the initial value problem

\[ y' + (\cos t)y^2 = 0, \quad y(0) = y_0, \]

where \( y_0 \) is an arbitrary constant.

(b) For what values of the initial data \( y_0 \) is your solution defined for all \(-\infty < t < +\infty\)?
3. [20 pts.] Consider the ordinary differential equation

\[ y' = (y - 3) (y^2 - 1) \]

(a) Sketch a graph of the right-hand side of this equation as a function of \( y \), and find all equilibrium solutions of the equation.

(b) Sketch the phase line of the equation, and determine the stability of the equilibria you found in (a).

(c) How does the solution with \( y(0) = 0 \) behave as \( t \to +\infty \)? How does the solution with \( y(0) = 2 \) behave as \( t \to -\infty \)?
4. [20 pts.] (a) Find the general solution of the equation
\[ y'' + y' - 2y = 0. \]
(b) Find the general solution of the equation
\[ y'' - 2y' + 5y = 0. \]
5. [20 pts.] Consider the nonhomogeneous ordinary differential equation

\[ y'' + 4y = 4t^2 + 10e^{-t}. \]

(a) Find a fundamental pair of solutions for the associated homogeneous equation

(b) Find a particular solution of the nonhomogeneous equation.

(c) Write out the general solution of the nonhomogeneous equation
6. [20 pts.] Consider the ordinary differential equation (in $t > 0$)

$$ty'' - (t + 1)y' + y = 0.$$ 

Note that $y_1(t) = e^t$ is a solution of this equation.

(a) Write $y(t) = v(t)e^t$ and derive an equation for $v(t)$.

(b) Solve the equation for $v(t)$ you derived in (a).

(c) Write out an expression for the general solution of the original ordinary differential equation.
7. [20 pts.] Find the general solution of the following $3 \times 3$ system:

$$\vec{x}'(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -14 & 7 \end{pmatrix} \vec{x}(t).$$
8. [20 pts.] Consider the ODE

\[ y'' - y = g(t), \]
\[ y(0) = 0, \quad y'(0) = 0, \]

where \( g(t) \) is a given continuous function.

(a) Write

\[ y(t) = e^t u_1(t) + e^{-t} u_2(t), \]

where

\[ e^t u'_1(t) + e^{-t} u'_2(t) = 0, \]

and solve for \( u_1(t), u_2(t) \). (This is the method of variation of parameters.)

(b) Give an expression for the general solution \( y(t) \) in terms of \( g(t) \).
9. [20 pts.] Find the general solution (expressed in terms of real-valued functions) of the following $2 \times 2$ system

$$\dot{x}(t) = \begin{pmatrix} -1 & 5 \\ -2 & -3 \end{pmatrix} x(t).$$
10. [20 pts.] Suppose that the $2 \times 2$ matrix $A$ has the following eigenvalues and eigenvectors:

$$r_1 = -3, \quad \xi_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \quad r_2 = -1, \quad \xi_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

(a) Sketch the trajectories of the system $\vec{x}'(t) = A\vec{x}(t)$, where $\vec{x} = (x_1, x_2)^T$, in the phase plane.

(b) On the next page, sketch the graphs of $x_1(t)$ and $x_2(t)$ versus $t$ for the solution that satisfies the initial condition $x_1(0) = 3, x_2(0) = 1$. 