Advanced Calculus
Math 25, Fall 2015
Final: Solutions

1. [20 pts] Say if the following statements are true or false. If false, give a counter-example, if true give a brief explanation why (a complete proof is not required).

(a) If \((a_n)\) is a sequence such that for every \(k \in \mathbb{N}\) there exists \(j \in \mathbb{N}\) such that \(a_j = a_{j+1} = \cdots = a_{j+k}\) (meaning that the sequence contains arbitrarily long strings of repeated terms), then \((a_n)\) converges.

(b) If the series \(\sum a_n\) is conditionally convergent, then the series \(\sum \sqrt{n}a_n\) diverges.

(c) If \(A \subset \mathbb{R}\) and every \(a \in A\) is an interior point of \(A\), then \(A\) is open.

(d) If \(A \subset \mathbb{R}\) and every \(a \in A\) is an isolated point of \(A\), then \(A\) is closed.

Solution.

- (a) False. For example, the sequence \((1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots)\) with \(n\) successive integers \(n\) does not converge.

- (b) False. For example, if \(a_n = (-1)^{n+1}/n\), then the alternating harmonic series \(\sum (-1)^{n+1}/n\) is conditionally convergent and \(\sum (-1)^{n+1}/\sqrt{n}\) converges by the alternating series test.

- (c) True. This follows immediately from the definitions. If \(a \in A\) is an interior point, then there exists \(\delta > 0\) such that \((a - \delta, a + \delta) \subset A\), so \(A\) is open if (and only if) every \(a \in A\) is an interior point.

- (d) False. For example, if \(A = \{1/n : n \in \mathbb{N}\}\), then every point of \(A\) is an isolated point, but \(A\) is not closed since \(0 \notin A\) is a limit point of \(A\).
2. [20 pts] Prove by induction that

\[(1 + 2 + 3 + \cdots + n)^2 = 1^3 + 2^3 + 3^3 + \cdots + n^3\]

for every natural number \(n \in \mathbb{N}\).

**Hint.** You can use the fact that the sum of the first \(n\) natural numbers is given by

\[1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1).\]

**Solution.**

- The result is true for \(n = 1\), since \(1^2 = 1^3\).

- Assume the result is true for some \(n \in \mathbb{N}\). Then, using the induction hypothesis and the sum given in the hint, we get that

\[
1^3 + 2^3 + 3 + \cdots + n^3 + (n + 1)^3 = (1 + 2 + 3 + \cdots + n)^2 + (n + 1)^3
\]

\[
= \left[\frac{1}{2}n(n + 1)\right]^2 + (n + 1)^3
\]

\[
= (n + 1)^2 \left[\frac{1}{4}n^2 + n + 1\right]
\]

\[
= \frac{1}{4}(n + 1)^2(n + 2)^2
\]

\[
= (1 + 2 + 3 + \cdots + n + n + 1)^2,
\]

so the result is true for \(n + 1\). It follows by induction that the result holds for every \(n \in \mathbb{N}\).

- Alternatively, you could prove by induction that

\[1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^2(n + 1)^2\]

and observe that the result follows from the sum in the hint.
3. [20 pts] Suppose that the sets $A, B \subset \mathbb{R}$ are bounded from above. Let

$$A + B = \{ x \in \mathbb{R} : x = a + b \text{ for some } a \in A \text{ and } b \in B \}.$$ 

Prove that $\sup(A + B) = \sup A + \sup B$.

Solution.

- Let $P = \sup A$, $Q = \sup B$, and $M = P + Q$. Since $P$ is an upper bound of $A$ and $Q$ is an upper bound of $B$, we have $a + b \leq P + Q$ for every $a \in A$ and $b \in B$, so $M$ is an upper bound of $A + B$.

- Suppose that $M' < M$ and let $\epsilon = M - M' > 0$. If $P' = P - \epsilon/2$ and $Q' = Q - \epsilon/2$, then $P' + Q' = M'$. Since $P$ is a least upper bound of $A$ and $P' < P$, there exists $a \in A$ such that $a > P'$; similarly, there exists $b \in B$ such that $b > Q'$. It follows that $a + b > M'$, so $M'$ is not an upper bound of $A + B$, which proves that $M$ is a least upper bound and $\sup(A + B) = \sup A + \sup B$. 

4. [20 pts] (a) State the definition of the convergence of a sequence \((a_n)\) of real numbers to a limit \(L\).

(b) Suppose that \(a_n \geq 0\) and \(\lim_{n \to \infty} a_n = 0\). Prove from the definition that \(\lim_{n \to \infty} \sqrt{a_n} = 0\).

Solution.

- (a) \(a_n \to L\) as \(n \to \infty\) if for every \(\epsilon > 0\) there exists \(N \in \mathbb{N}\) such that \(n > N\) implies that \(|a_n - L| < \epsilon\).

- (b) Let \(\epsilon > 0\). Since \(a_n \to 0\), there exists \(N \in \mathbb{N}\) such that \(n > N\) implies that \(0 \leq a_n < \epsilon^2\). It follows that \(n > N\) implies that \(\sqrt{a_n} < \epsilon\), which proves that \(\sqrt{a_n} \to 0\) as \(n \to \infty\).
5. [20 pts] Let \((a_n)\) be a bounded sequence of real numbers.
(a) State the definition of \(\limsup_{n \to \infty} a_n\).
(b) Prove that there is a subsequence \((a_{n_k})\) of \((a_n)\) such that
\[
\lim_{k \to \infty} a_{n_k} = \limsup_{n \to \infty} a_n.
\]
(c) Prove that if \((a_{n_k})\) is any convergent subsequence of \((a_n)\), then
\[
\lim_{k \to \infty} a_{n_k} \leq \limsup_{n \to \infty} a_n.
\]
Solution.
- (a) \[
\limsup_{n \to \infty} a_n = \lim_{n \to \infty} b_n, \quad b_n = \sup\{a_k : k \geq n\}.
\]
- (b) Let \(L = \limsup_{n \to \infty} a_n\), so \(b_n \downarrow L\). For each \(k \in \mathbb{N}\) there exists \(N_k \in \mathbb{N}\) such that \(n > N_k\) implies that
\[
L \leq b_n < L + \frac{1}{k+1}.
\]
We construct a subsequence \((a_{n_k})\) recursively as follows. First, choose any \(n_1 \in \mathbb{N}\). Then, given \(n_k\) for \(k \in \mathbb{N}\), choose some \(n > \max\{n_k, N_k\}\). By the definition of the supremum that defines \(b_n\), there exists \(n_{k+1} \geq n\) such that
\[
b_n - \frac{1}{k+1} \leq a_{n_{k+1}} \leq b_n.
\]
It follows that \(n_{k+1} > n_k\) and
\[
L - \frac{1}{k+1} < a_{n_{k+1}} < L + \frac{1}{k+1},
\]
or \(|a_{n_k} - L| < 1/k\), for every \(k \geq 2\). The sandwich theorem then implies that \(a_{n_k} \to L\) as \(k \to \infty\).
- (c) Suppose that \((a_{n_k})\) is a convergent subsequence of \((a_n)\). For every \(k \in \mathbb{N}\), we have \(a_{n_k} \leq b_{n_k}\). Taking the limit of this inequality as \(k \to \infty\), then using the monotonicity property of limits and the fact that every subsequence of \((b_n)\) converges to the same limit as \((b_n)\), we get that
\[
\lim_{k \to \infty} a_{n_k} \leq \limsup_{n \to \infty} a_n.
\]
6. [20 pts] Determine the convergence of the following series. Justify your answers.

(a) \[ \sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}!} \]

(b) \[ \sum_{n=1}^{\infty} \frac{n-1}{n^2+1} \]

(c) \[ \sum_{n=1}^{\infty} (-1)^{n+1} \left( \frac{4^n - 3}{n^4 + 3} \right) \]

(d) \[ \sum_{n=1}^{\infty} \left( \frac{n-1}{n} - \frac{n}{n+1} \right) \]

Solution.

- (a) If \( n \geq 16 \), then \( \frac{2}{\sqrt{n}} \leq \frac{1}{2} \), and it follows that
  \[ \frac{2^n}{\sqrt{n}!} = \frac{2}{\sqrt{1}} \cdot \frac{2}{\sqrt{2}} \cdot \frac{2}{\sqrt{3}} \cdots \frac{2}{\sqrt{n}} \leq C \frac{2^n}{2^n} \]
  for a suitable constant \( C \) that is independent of \( n \). So the series converges absolutely by comparison with a convergent geometric series.

- (b) For \( n \geq 2 \), we have \( n - 1 \geq n/2 \) and
  \[ \frac{n-1}{n^2+1} > \frac{n/2}{n^2+n^2} = \frac{1}{4n} \]
  It follows that the series diverges to \( \infty \) by comparison with the divergent harmonic series.

- (c) The series diverges because \( \frac{4^n}{n^4} \to \infty \) as \( n \to \infty \), so its terms diverge to \( \infty \). To prove this limit, note that for \( n \geq 6 \) we have
  \[ 4^n = (1 + 3)^n > \left( \frac{n}{4} \right)^{3n-4} = \frac{n(n-1)(n-2)(n-3)3^n}{4!3^4} > \frac{n^43^n}{4!3^42^3} \]
  where we retain only one term in the binomial expansion of \( (1 + 3)^n \), and \( 3^n \to \infty \) as \( n \to \infty \).

- (d) This is a telescoping series of negative terms, and
  \[ \sum_{n=1}^{N} \left| \frac{n-1}{n} - \frac{n}{n+1} \right| = \sum_{n=1}^{N} \left( \frac{n}{n+1} - \frac{n-1}{n} \right) = \frac{N}{N+1} - 0 \to 1 \]
  as \( N \to \infty \), so the series converges absolutely (to \(-1\)).
7. [20 pts] (a) Define an open set $G \subset \mathbb{R}$.

(b) If $\{G_1, G_2, \ldots, G_n\}$ is a finite collection of open sets $G_i \subset \mathbb{R}$, prove that $G = \bigcap_{i=1}^n G_i$ is open.

(c) If $\{G_i : i \in \mathbb{N}\}$ is an infinite collection of open sets $G_i \subset \mathbb{R}$, give an example to show that that $\bigcap_{i=1}^\infty G_i$ need not be open.

Solution.

• (a) A set $G \subset \mathbb{R}$ is open if for every $x \in G$ there exists $\delta > 0$ such that $(x - \delta, x + \delta) \subset G$.

• (b) Suppose that $x \in \bigcap_{i=1}^n G_i$. Then $x \in G_i$ for every $1 \leq i \leq n$. Since $G_i$ is open, there exists $\delta_i > 0$ such that $(x - \delta_i, x + \delta_i) \subset G_i$. If

$$\delta = \min\{\delta_i : 1 \leq i \leq n\},$$

then $\delta > 0$ and $(x - \delta, x + \delta) \subset G_i$ for every $i$, so $(x - \delta, x + \delta) \subset \bigcap_{i=1}^n G_i$, which proves that $\bigcap_{i=1}^n G_i$ is open.

• (c) Let

$$G_i = \left(0, 1 + \frac{1}{i}\right)$$

Then $G_i$ is open but

$$\bigcap_{i=1}^\infty G_i = (0, 1]$$

is not open.